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On the higher order asymptotic analysis of a non-uniformly propagating dynamic crack along an arbitrary path

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Abstract. Transient mixed-mode elastodynamic crack growth along arbitrary smoothly varying paths is considered. Asymptotically, the crack tip stress field is square root singular with the angular variation of the singular term depending weakly on the instantaneous values of the crack tip speed and on the mode-I and mode-II stress intensity factors. However, for a material particle at a small distance away from the moving crack tip, the local stress field will depend not only on the instantaneous values of the crack tip speed and stress intensity factors, but also on the past history of these time dependent quantities. In addition, for cracks propagating along curved paths the stress field is also expected to depend on the nature of the curved crack path. Here, a representation of the crack tip fields in the form of an expansion about the crack tip is obtained in powers of radial distance from the tip. The higher order coefficients of this expansion are found to depend on the time derivative of crack tip speed, the time derivatives of the two stress intensity factors as well as on the instantaneous value of the local curvature of the crack path. It is also demonstrated that even if cracks follow a curved path dictated by the criterion $K_{II}^{d} = 0$, the stress field may still retain higher order asymmetric components related to non-zero local curvature of the crack path.

Key words: Dynamic fracture, crack propagation and curving, transient effects, asymptotic analysis.

1. Introduction

Since Irwin [1] observed that the elastic stress field near the tip of a static crack has a universal spatial structure, and the magnitude of the stress field is controlled by a scalar quantity, the elastic stress intensity factor, this quantity has played one of the most important roles in linear elastic fracture mechanics. For propagating cracks, the early analytic results of Yoffe [2], Craggs [3], Broberg [4], Baker [5] and Freund [6], among others, revealed that the asymptotic stress field near the moving crack tip has a universal structure as well. As stated by Freund and Clifton [7], the stress field with reference to a Cartesian coordinate system moving with the crack tip of "all plane elasto-dynamic solutions for (smoothly turning) running cracks, for which the total internal energy is finite," can be asymptotically described by the square root singular expression,

$$\sigma_{\alpha\beta} = \frac{K_I^d(t)}{\sqrt{2\pi r}} \Sigma_{\alpha\beta}^I(\theta, v) + \frac{K_{II}^d(t)}{\sqrt{2\pi r}} \Sigma_{\alpha\beta}^{II}(\theta, v) + O(1), \quad \text{as } r \to 0. \tag{1}$$

Here (r, θ) is a polar coordinate system traveling with the crack tip, $\Sigma_{\alpha\beta}^I(\theta, v)$ and $\Sigma_{\alpha\beta}^{II}(\theta, v)$ are known universal functions of θ and crack tip speed v, and $K_I^d(t)$ and $K_{II}^d(t)$ are the mode-I and mode-II stress intensity factors, respectively. They are dependent only on the specific geometric and loading conditions of a problem. In addition to the most singular asymptotic representation of the stress field, Nishioka and Atluri [8] and Dally [9] also developed the entire higher order asymptotic expression for the stress field near the tip of a dynamically moving straight crack under *steady state* conditions.

Expression (1) is strictly valid only in the immediate vicinity of the crack tip. To apply this expression over a region of finite extent, one must show that the asymptotic solution indeed dominates over this region, and this domain is then referred to as a region of K^d-dominance. Recent experimental evidences obtained by means of optical techniques, e.g. the method of caustics (Krishnaswamy and Rosakis [10] and Rosakis et al. [11]) and the Coherent Gradient Sensing technique (CGS) (Krishnaswamy et al. [12]), have shown that the assumption of K^d-dominance is often violated during the process of dynamic fracture, and that the expression in (1) is insufficient to characterize the deformation field near the crack tip. It was observed that the violation of the assumption of K_I^d-dominance is often associated with the existence of highly transient crack growth motions involving crack tip accelerations as well as fast varying stress intensity factor histories, evens that are typical of most laboratory dynamic testing situations.

By using the asymptotic methodology introduced by Freund [13], and by relaxing the assumption of K_I^d-dominance, Freund and Rosakis [14] have provided a higher order asymptotic expansion for the first stress invariant (quantity of interest for both caustics and CGS) and showed that this expansion provides an accurate description of crack tip fields under fairly severe transient conditions. Later, Rosakis et al. [15] obtained the higher order asymptotic stress field near the tip of a non-uniformly propagating mode-I crack. In a related study, Liu et al. [16] have also applied these results to the interpretation of optical caustic patterns and have confirmed the advantages of the higher order expansion in analyzing experimental data.

This paper represents the natural continuation of the studies discussed above. Our purpose is to understand the nature of the mixed mode asymptotic field that dominates the region near a transiently propagating and curving crack tip. In this paper, we develop a new methodology to obtain the higher order transient asymptotic elastodynamic field near the tip of a crack that propagates non-uniformly along an arbitrary and smoothly curved path. Here, we consider crack growth in a homogeneous, isotropic, and linearly elastic material. The deformation is assumed to be plane strain. However for plane stress similar results can be obtained by changing the expression for some material parameters. By using the asymptotic procedure proposed by Freund [13] and utilized by Freund and Rosakis [14], the governing equation is reduced to a series of coupled partial differential equations, and the problem can be further recast into a Riemann–Hilbert problem. Upon solving the Riemann–Hilbert equation, the higher order near-tip transient elastodynamic asymptotic field can be obtained. The results show that the singular terms and the so-called T-stress term have the same spatial form as those obtained under steady state conditions. However, the dynamic stress intensity factors and the crack tip velocity are now allowed to be functions of time. The third term, on the other hand, depends not only on the instantaneous values of the crack tip speed and the stress intensity factors, but also on the past history of these time-dependent quantities (i.e. on $\dot{K}_I^d(t)$, $\dot{K}_{II}^d(t)$, and $\dot{v}(t)$). For a crack that propagates along a curved path, the third term also depends on the curvature of the crack path at the crack tip. Some implications of these analytic results on the interpretation of experimental observations of crack curving are also discussed.

2. General formulation

Consider a planar body composed of homogeneous, isotropic, linearly elastic material. In the body, there is an arbitrarily propagating crack. Introduce a fixed orthonormal Cartesian coordinate system (x_1, x_2) so that at a time $t = 0$, the crack tip happens to be at the origin of the system. For any $t > 0$, the position of the propagating crack tip is supposed to be given by $(X_1(t), X_2(t))$, see Fig. 1. If the deformation is plane strain, we may consider the two displacement potentials, $\phi(x_1, x_2, t)$ and $\psi(x_1, x_2, t)$, and then the two non-zero displacement components can be expressed by

$$u_\alpha(x_1, x_2, t) = \phi_{,\alpha}(x_1, x_2, t) + e_{\alpha\beta}\psi_{,\beta}(x_1, x_2, t), \tag{2}$$

where α, $\beta \in \{1, 2\}$ and the summation convention is employed here. $e_{\alpha\beta}$ is the two-dimensional alternative symbol and is defined by

$$e_{12} = -e_{21} = 1, \qquad e_{11} = e_{22} = 0.$$

The components of stress for the material we consider can be expressed by the

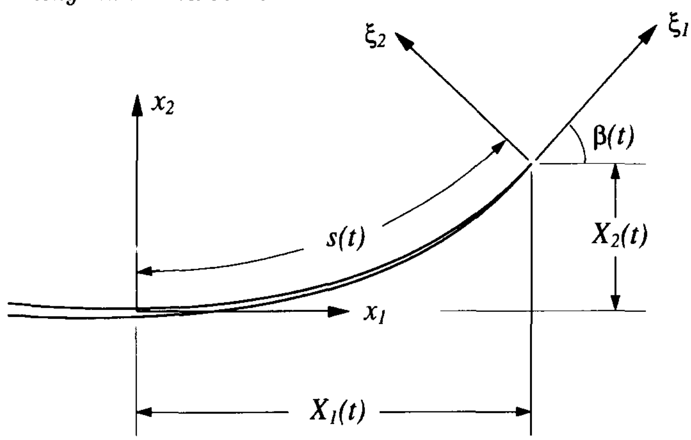

Fig. 1. Crack growing along a smooth curved path under two-dimensional conditions. The instantaneous crack tip position is $x_1 = X_1(t)$, $x_2 = X_2(t)$, and the instantaneous crack tip speed is $v(t)$ in the local ξ_1-direction.

displacement potentials like

$$\sigma_{11} = \mu \left\{\frac{c_l^2}{c_s^2}\,\phi_{,\alpha\alpha} - 2\phi_{,22} + 2\psi_{,12}\right\}$$

$$\sigma_{22} = \mu \left\{\frac{c_l^2}{c_s^2}\,\phi_{,\alpha\alpha} - 2\phi_{,11} - 2\psi_{,12}\right\}$$

$$\sigma_{12} = \mu\{2\phi_{,12} + \psi_{,22} - \psi_{,11}\} \tag{3}$$

where μ is the shear modulus, and c_l, c_s are the longitudinal and shear wave speeds of the elastic material, respectively. In terms of the shear modulus μ, c_l and c_s are given by

$$c_l = \left\{\frac{\kappa+1}{\kappa-1}\cdot\frac{\mu}{\rho}\right\}^{1/2}, \qquad c_s = \left\{\frac{\mu}{\rho}\right\}^{1/2}, \tag{4}$$

where $\kappa = 3-4\nu$ for plane strain and $\kappa = (3-\nu)/(1+\nu)$ for plane stress, and ρ is the mass density of the material, ν is the Poisson's ratio. By changing the definition for the longitudinal wave speed in Eq. (4), the solution corresponding to the plane stress deformation can be obtained. Meanwhile, c_l and c_s in both plane strain and plane stress, are related by

$$\frac{c_s}{c_l} = \left\{\frac{\kappa-1}{\kappa+1}\right\}^{1/2}. \tag{5}$$

The equation of motion in the absence of body force in the fixed coordinate system, in terms of $\phi(x_1, x_2, t)$ and $\psi(x_1, x_2, t)$ is

$$\phi_{,\alpha\alpha}(x_1, x_2, t) - \frac{1}{c_l^2}\,\ddot{\phi}(x_1, x_2, t) = 0$$

$$\psi_{,\alpha\alpha}(x_1, x_2, t) - \frac{1}{c_s^2}\,\ddot{\psi}(x_1, x_2, t) = 0. \tag{6}$$

Now introduce a new moving coordinate system, (ξ_1, ξ_2), so that the origin of the new system is at the moving crack tip. The ξ_1-axis is tangential to the crack trajectory at the crack tip and coincides with the direction of the crack growth. The angle between the ξ_1-axis and the fixed x_1-axis is denoted by $\beta(t)$, as shown in Fig. 1. Therefore, the relation between the coordinates in these two systems is

$$\xi_1 = \{x_1 - X_1(t)\}\,\cos\beta(t) + \{x_2 - X_2(t)\}\,\sin\beta(t)$$

$$\xi_2 = -\{x_1 - X_1(t)\}\,\sin\beta(t) + \{x_2 - X_2(t)\}\,\cos\beta(t). \tag{7}$$

In this new system, the equation of motion (6) for $\phi(\xi_1, \xi_2, t)$ and $\psi(\xi_1, \xi_2, t)$ will be [13]

$$\phi_{,\alpha\alpha} - \frac{1}{c_l^2}\,\{\phi_{,\alpha\beta}\dot{\xi}_\alpha\dot{\xi}_\beta + \phi_{,\alpha}\ddot{\xi}_\alpha + 2\phi_{,\alpha t}\dot{\xi}_\alpha + \phi_{,tt}\} = 0$$

$$\psi_{,\alpha\alpha} - \frac{1}{c_s^2}\,\{\psi_{,\alpha\beta}\dot{\xi}_\alpha\dot{\xi}_\beta + \psi_{,\alpha}\ddot{\xi}_\alpha + 2\psi_{,\alpha t}\dot{\xi}_\alpha + \psi_{,tt}\} = 0. \tag{8}$$

If the length of the trajectory that the crack tip travels during the time interval $[0, t]$, is denoted by $s(t)$, then the magnitude of the crack tip speed $v(t)$ will be $\dot{s}(t)$, and the curvature of the crack trajectory at the crack tip, $k(t)$, is given by

$$k(t) = \frac{d\beta}{ds} = \frac{\dot{\beta}(t)}{v(t)}. \tag{9}$$

In terms of the crack tip speed $v(t)$, and the crack tip curvature $k(t)$, we have the relation

$$\dot{\xi}_1 = -v(t) + v(t)k(t)\xi_2, \qquad \dot{\xi}_2 = -v(t)k(t)\xi_1. \tag{10}$$

As a result of Eq. (10), we can also express the $\ddot{\xi}_\alpha$ in terms of the crack tip speed and the crack tip curvature. Now, the equation of motion (8) can be rewritten as

$$
\begin{aligned}
&\phi_{,11} + \frac{1}{\alpha_l^2}\,\phi_{,22} + \frac{2\sqrt{v}}{\alpha_l^2 c_l^2}\,\{\sqrt{v}\phi_{,1}\}_{,t} - \frac{1}{\alpha_l^2 c_l^2}\,\phi_{,tt} \\
&-\frac{v^2 k}{\alpha_l^2 c_l^2}\,\{\phi_{,2} + 2\xi_1\phi_{,12} - 2\xi_2\phi_{,11}\} - \frac{2\sqrt{vk}}{\alpha_l^2 c_l^2}\,\{\sqrt{vk}(\xi_2\phi_{,1} - \xi_1\phi_{,2})\}_{,t} \\
&+\frac{v^2 k^2}{\alpha_l^2 c_l^2}\,\{\xi_2^2\phi_{,11} - 2\xi_1\xi_2\phi_{,12} + \xi_1^2\phi_{,22} - \xi_1\phi_{,1} - \xi_2\phi_{,2}\} = 0,
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
&\psi_{,11} + \frac{1}{\alpha_s^2}\,\psi_{,22} + \frac{2\sqrt{v}}{\alpha_s^2 c_s^2}\,\{\sqrt{v}\psi_{,1}\}_{,t} - \frac{1}{\alpha_s^2 c_s^2}\,\psi_{,tt} \\
&-\frac{\sqrt{v}\,k}{\alpha_s^2 c_s^2}\,\{\psi_{,2} + 2\xi_1\psi_{,12} - 2\xi_2\psi_{,11}\} - \frac{2\sqrt{vk}}{\alpha_s^2 c_s^2}\,\{\sqrt{vk}(\xi_2\psi_{,1} - \xi_1\psi_{,2})\}_{,t}, \\
&+\frac{v^2 k^2}{\alpha_s^2 c_s^2}\,\{\xi_2^2\psi_{,11} - 2\xi_1\xi_2\psi_{,12} + \xi_1^2\psi_{,22} - \xi_1\psi_{,1} - \xi_2\psi_{,2}\} = 0,
\end{aligned}
\tag{12}
$$

where the two quantities α_l and α_s depend on the crack tip speed, and therefore depend on time t through

$$
\alpha_{l,s}(t) = \left\{1 - \frac{v^2(t)}{c_{l,s}^2}\right\}^{1/2}.
$$

Notice that in Eqs. (11) and (12), the derivative with respect to time, t, is distinct from that in Eq. (6). Here, ξ_1, ξ_2 are held fixed, whereas in (6), x_1, x_2 are held fixed. Throughout this study, we will use $\partial/\partial t$, or $\{\ \}_{,t}$ to denote the differentiation with respect to time, t, where the moving spatial coordinates are held fixed, while using $\{\dot{}\}$ denote the same operation but the fixed spatial coordinates are held fixed.

At this point, we employ the standard asymptotic device used by Freund and Rosakis [14] for the analysis of transient mode-I crack growth. We assume that $\phi(\xi_1, \xi_2, t)$ and $\psi(\xi_1, \xi_2, t)$ can be asymptotically expanded as

$$
\begin{aligned}
\phi(\xi_1, \xi_2, t) &= \sum_{m=0}^{\infty} \varepsilon^{p_m}\phi_m(\eta_1, \eta_2, t) \\
\psi(\xi_1, \xi_2, t) &= \sum_{m=0}^{\infty} \varepsilon^{p_m}\psi_m(\eta_1, \eta_2, t),
\end{aligned}
\tag{13}
$$

as $r = (\xi_1^2 + \xi_2^2)^{1/2} \to 0$, where $\eta_\alpha = \xi_\alpha/\varepsilon$, $\alpha \in \{1, 2\}$, and ε is a small arbitrary positive number. The parameter ε is used here so that the region around the crack tip is expanded to fill the entire field of observation. As ε is chosen to be infinitely small, all points in the (ξ_1, ξ_2) plane except those very close to the crack tip, are pushed out of the field of observation in the (η_1, η_2) plane. If the trajectory of the moving crack is smooth enough, the crack line will occupy the entire negative η_1-axis in this scaled plane. By taking $\varepsilon = 1$, the above equation will provide the asymptotic representation of the displacement potentials in the unscaled physical plane.

In the asymptotic representation (13), the powers of ε are

$$p_{m+1} = p_m + \tfrac{1}{2}, \qquad m = 0, 1, 2, \ldots, \tag{14}$$

so that the nontrivial solutions for $\phi_m(\eta_1, \eta_2, t)$ exist. Since the displacement should be bounded throughout the region, but the stress may be singular at the crack tip, p_0 is expected to be in the range $1 < p_0 < 2$. We also should have that

$$\frac{\varepsilon^{p_{m+n}}\phi_{m+n}(\eta_1,\ \eta_2,\ t)}{\varepsilon^{p_m}\phi_m(\eta_1,\ \eta_2,\ t)} \to 0, \quad \text{as } \varepsilon \to 0, \tag{15}$$

for any positive integer n. Meanwhile, as we return to the physical plane, we will have

$$\frac{\phi_{m+n}(\xi_1,\ \xi_2,\ t)}{\phi_m(\xi_1,\ \xi_2,\ t)} \to 0, \quad \text{as } r = \sqrt{\xi_1^2 + \xi_2^2} \to 0, \tag{16}$$

for any positive integer n, so that in the physical plane, (ξ_1, ξ_2), $\phi_m(\xi_1, \xi_2, t)$ are ordered according to their contributions to the near tip deformation field. The above properties for ϕ_m hold for ψ_m as well.

Substituting the asymptotic representations for $\phi(\xi_1,\ \xi_2,\ t)$ and $\psi(\xi_1,\ \xi_2,\ t)$, Eq. (13), into the equations of motion, (11) and (12), we will obtain two equations where the left-hand side is an infinite power series of ε. Since ε is an arbitrary number, the coefficient of each power of ε should be zero. Therefore, the equations of motion reduce to a series of coupled differential equations for $\phi_m(\eta_1, \eta_2, t)$ and $\psi_m(\eta_1, \eta_2, t)$ as follows

$$\phi_{m,11} + \frac{1}{\alpha_l^2}\,\phi_{m,22} = -\frac{2\sqrt{v}}{\alpha_l^2 c_l^2}\,\{\sqrt{v}\,\phi_{m-2,1}\}_{,t} + \frac{1}{\alpha_l^2 c_l^2}\,\phi_{m-4,tt}$$
$$+ \frac{(1-\alpha_l^2)k}{\alpha_l^2}\,\{\phi_{m-2,2} + 2\eta_1\phi_{m-2,12} - 2\eta_2\phi_{m-2,11}\}$$

$$+\frac{2\sqrt{vk}}{\alpha_l^2 c_l^2}\{\sqrt{vk}(\eta_2\phi_{m-4,1}-\eta_1\phi_{m-4,2})\}_{,t}+\frac{(1-\alpha_l^2)k^2}{\alpha_l^2}\{\eta_2^2\phi_{m-4,11}$$

$$-2\eta_1\eta_2\phi_{m-4,12}+\eta_1^2\phi_{m-4,22}-\eta_1\phi_{m-4,1}-\eta_2\phi_{m-4,2}\}, \qquad (17)$$

and

$$\psi_{m,11}+\frac{1}{\alpha_s^2}\psi_{m,22}=-\frac{2\sqrt{v}}{\alpha_s^2 c_s^2}\{\sqrt{v}\psi_{m-2,1}\}_{,t}+\frac{1}{\alpha_s^2 c_s^2}\psi_{m-4,tt}$$

$$+\frac{(1-\alpha_s^2)k}{\alpha_s^2}\{\psi_{m-2,2}+2\eta_1\psi_{m-2,12}-2\eta_2\psi_{m-2,11}\}$$

$$+\frac{2\sqrt{vk}}{\alpha_s^2 c_s^2}\{\sqrt{vk}(\eta_2\psi_{m-4,1}-\eta_1\psi_{m-4,2})\}_{,t}+\frac{(1-\alpha_s^2)k^2}{\alpha_s^2}\{\eta_2^2\psi_{m-4,11}$$

$$-2\eta_1\eta_2\psi_{m-4,12}+\eta_1^2\psi_{m-4,22}-\eta_1\psi_{m-4,1}-\eta_2\psi_{m-4,2}\}, \qquad (18)$$

for $m = 0, 1, 2, \ldots,$ and where

$$\phi_m=\begin{cases}\phi_m & \text{for } m\geqslant 0\\ 0 & \text{for } m<0\end{cases}, \qquad \psi_m=\begin{cases}\psi_m & \text{for } m\geqslant 0\\ 0 & \text{for } m<0\end{cases}. \qquad (19)$$

It is noted that, for a crack propagating along a straight trajectory, $k(t) = 0$, and Eqs. (17) and (18) reduce to that given by Rosakis et al. [15]. The term "coupled" is used above in the sense that ϕ_m or ψ_m with higher values of m will be affected by the solutions for ϕ_m or ψ_m with lower vlues of m. Furthermore, for the special case of *steady state* crack growth, the crack tip velocity, v, will be a constant, and at the same time, $\phi_{m,t} = \psi_{m,t} = 0$, for $m = 0, 1, 2, \ldots,$ which means that ϕ_m and ψ_m depend on t only through the spatial coordinate η_1. In such a case, the equations in (17) and (18) are not coupled anymore and each one reduces to Laplace's equation in the coordinates $(\eta_1, \alpha_l\eta_2)$ for ϕ_m and $(\eta_1, \alpha_s\eta_2)$ for ψ_m, respectively. The corresponding functions ϕ_m and ψ_m are independent of time in the moving coordinate system. The solution for this case is discussed by Dally [9] who attributes the original results to G.R. Irwin. However, for the transient case, the crack may propagate along an arbitrary path, the crack tip velocity, $v(t)$, may be a continuous function of time and so is the crack tip curvature. Also, $\phi_m(\eta_1, \eta_2, t)$ and $\psi_m(\eta_1, \eta_2, t)$ may depend on time explicitly in the moving coordinate system. The only uncoupled equations are those for $m = 0$ and $m = 1$. As $m > 1$, we can see from Eqs. (17) and (18) that $\phi_m(\eta_1, \eta_2, t)$, or $\psi_m(\eta_1, \eta_2, t)$ is composed by two parts, one is the particular solution which is completely determined by the previous terms, the other part is the homogeneous solution which satisfies the Laplace's equation in the corresponding scaled coordinate plane. Suppose that there is no traction applied on the crack faces, then the combination of the particular and homogeneous solutions should satisfy the traction free condition on the crack faces. In the following section, we will solve $\phi_m(\eta_1, \eta_2, t)$ and $\psi_m(\eta_1, \eta_2, t)$ for the most general transient situation of a crack propagating along an arbitrary path.

3. Solution for the higher order transient problem

As we have discussed in the previous section, in Eqs. (17) and (18), the only uncoupled equations are those for $m = 0$ and $m = 1$. As $m > 1$, the solutions for $\phi_m(\eta_1, \eta_2, t)$ and $\psi_m(\eta_1, \eta_2, t)$ will be affected by the solutions with smaller m. Thus, in this section, we consider the situation of $m = 0$ and $m = 1$ first. After we get solutions for $m = 0$ and 1, we will subsequently solve $\phi_m(\eta_1, \eta_2, t)$ and $\psi_m(\eta_1, \eta_2, t)$ for higher order terms.

3.1. Solutions for $\phi_m(\eta_1, \eta_2, t)$ and $\psi_m(\eta_1, \eta_2, t)$ for $m = 0$ and 1

For $m = 0$, or 1, the equations of motion (17) and (18) reduce to

$$\phi_{m,11}(\eta_1, \eta_2, t) + \frac{1}{\alpha_l^2(t)}\, \phi_{m,22}(\eta_1, \eta_2, t) = 0$$

$$\psi_{m,11}(\eta_1, \eta_2, t) + \frac{1}{\alpha_s^2(t)}\, \psi_{m,22}(\eta_1, \eta_2, t) = 0. \qquad (20)$$

They are Laplace's equations in the corresponding scaled plane $(\eta_1, \alpha_l(t)\eta_2)$ for ϕ_m, and $(\eta_1, \alpha_s(t)\eta_2)$ for ψ_m. The most general solutions for Eq. (20) can be expressed as

$$\phi_m(\eta_1, \eta_2, t) = \mathrm{Re}\{F_m(z_l; t)\}$$

$$\psi_m(\eta_1, \eta_2, t) = \mathrm{Im}\{G_m(z_s; t)\}, \qquad (21)$$

where the complex variables z_l and z_s are given by

$$z_l = \eta_1 + i\alpha_l\eta_2, \qquad z_s = \eta_1 + i\alpha_s\eta_2,$$

and $i = \sqrt{-1}$. $F_m(z_l; t)$ and $G_m(z_s; t)$ are analytic everywhere in the complex z_l-, or z_s-planes except along the nonpositive real axes. In the analytic functions $F_m(z_l; t)$ and $G_m(z_s; t)$, the time t appears as a parameter. This suggests that $\phi_m(\eta_1, \eta_2, t)$ and $\psi_m(\eta_1, \eta_2, t)$ may depend on time t not only through the complex variables, z_l and z_s, but also explicitly through the time t itself.

Associated with these ϕ_m and ψ_m, the contributions to the displacement and stress components are given by

$$\begin{aligned} u_1^{(m)} &= \mathrm{Re}\{F'_m(z_l; t) + \alpha_s G'_m(z_s; t)\} \\ u_2^{(m)} &= -\mathrm{Im}\{\alpha_l F'_m(z_l; t) + G'_m(z_s; t)\}, \end{aligned} \tag{22}$$

and

$$\begin{aligned} \sigma_{11}^{(m)} &= \mu\,\mathrm{Re}\{(1 + 2\alpha_l^2 - \alpha_s^2)F''_m(z_l; t) + 2\alpha_s G''_m(z_s; t)\} \\ \sigma_{22}^{(m)} &= -\mu\,\mathrm{Re}\{(1 + \alpha_s^2)F''_m(z_l; t) + 2\alpha_s G''_m(z_s; t)\} \\ \sigma_{12}^{(m)} &= -\mu\,\mathrm{Im}\{2\alpha_l F''_m(z_l; t) + (1 + \alpha_s^2)G''_m(z_s; t)\}, \end{aligned} \tag{23}$$

where the prime represents the derivative with respect to the corresponding complex argument.

Denote that

$$\left.\begin{aligned} \lim_{\eta_2 \to 0^+} \Omega(z) &= \Omega^+(\eta_1) \\ \lim_{\eta_2 \to 0^-} \Omega(z) &= \Omega^-(\eta_1) \end{aligned}\right\}, \qquad z = \eta_1 + i\eta_2.$$

As $\eta_1 < 0$ and $\eta_2 \to 0^\pm$, the traction free condition on the crack faces implies that $\sigma_{22}^{(m)}(\eta_1, 0^\pm, t) = \sigma_{12}^{(m)}(\eta_1, 0^\pm, t) = 0$, or, in terms of the complex displacement potentials, $F_m(z_l; t)$ and $G_m(z_s; t)$,

$$\left.\begin{aligned} \mu(1+\alpha_s^2)\{F''^{\pm}_m(\eta_1; t) + \bar{F}''^{\mp}_m(\eta_1; t)\} + 2\mu\alpha_s\{G''^{\pm}_m(\eta_1; t) + \bar{G}''^{\mp}_m(\eta_1; t)\} = 0 \\ 2\mu\alpha_l\{F''^{\pm}_m(\eta_1; t) - \bar{F}''^{\mp}_m(\eta_1; t)\} + \mu(1+\alpha_s^2)\{G''^{\pm}_m(\eta_1; t) - \bar{G}''^{\mp}_m(\eta_1; t)\} = 0 \end{aligned}\right\}, \quad \forall \eta_1 < 0, \tag{24}$$

where the overline stands for the complex conjugate. Here, it seems that we have four unknown functions, $F_m(z_l; t)$, $\bar{F}_m(z_l; t)$, $G_m(z_s; t)$, and $\bar{G}_m(z_s; t)$, while we only have two independent relations in Eq. (24). However, these four functions can be related by the fact that the displacement components and the traction components should be continuous when they cross the real axis ahead of the crack tip, or in terms of the complex displacement potentials, $F_m(z_l;t)$ and $G_m(z_s;t)$, along $\eta_1 > 0$ and $\eta_2 = 0$, we should have

$$\left.\begin{aligned}
&\mu(1+\alpha_s^2)\{F_m''^{+}(\eta_1;t)+\bar{F}_m''^{-}(\eta_1;t)\}\\
&\quad+2\mu\alpha_s\{G_m''^{+}(\eta_1;t)+\bar{G}_m''^{-}(\eta_1;t)\}\\
&\quad-\mu(1+\alpha_s^2)\{F_m''^{-}(\eta_1;t)+\bar{F}_m''^{+}(\eta_1;t)\}\\
&\quad-2\mu\alpha_s\{G_m''^{-}(\eta_1;t)+\bar{G}_m''^{+}(\eta_1;t\}=0\\
&2\mu\alpha_l\{F_m''^{+}(\eta_1;t)-\bar{F}_m''^{-}(\eta_1;t)\}\\
&\quad+\mu(1+\alpha_s^2)\{G_m''^{+}(\eta_1;t)-\bar{G}_m''^{-}(\eta_1;t)\}\\
&\quad-2\mu\alpha_l\{F_m''^{-}(\eta_1;t)-\bar{F}_m''^{+}(\eta_1;t)\}\\
&\quad-\mu(1+\alpha_s^2)\{G_m''^{-}(\eta_1;t)-\bar{G}_m''^{+}(\eta_1;t)\}=0,
\end{aligned}\right\},\quad \forall\eta_1>0, \tag{25}$$

and

$$\left.\begin{aligned}
&\{F_m'^{+}(\eta_1;t)+\bar{F}_m'^{-}(\eta_1;t)\}+\alpha_s\{G_m'^{+}(\eta_1;t)+\bar{G}_m'^{-}(\eta_1;t)\}\\
&\quad-\{F_m'^{-}(\eta_1;t)+\bar{F}_m'^{+}(\eta_1;t)\}-\alpha_s\{G_m'^{-}(\eta_1;t)+\bar{G}_m'^{+}(\eta_1;t)\}=0\\
&\alpha_l\{F_m'^{+}(\eta_1;t)-\bar{F}_m'^{-}(\eta_1;t)\}+\{G_m'^{+}(\eta_1;t)-\bar{G}_m'^{-}(\eta_1;t)\}\\
&\quad-\alpha_l\{F_m'^{-}(\eta_1;t)-\bar{F}_m'^{+}(\eta_1;t)\}-\{G_m'^{-}(\eta_1;t)-\bar{G}_m'^{+}(\eta_1;t)\}=0
\end{aligned}\right\},\quad \forall\eta_1>0. \tag{26}$$

For simplicity, define the following matrices

$$\mathbf{P}=\begin{bmatrix}\mu(1+\alpha_s^2) & 2\mu\alpha_s\\ 2\mu\alpha_l & \mu(1+\alpha_s^2)\end{bmatrix},\qquad \mathbf{Q}=\begin{bmatrix}\mu(1+\alpha_s^2) & 2\mu\alpha_s\\ -2\mu\alpha_l & -\mu(1+\alpha_s^2)\end{bmatrix},$$

and

$$\mathbf{U}=\begin{bmatrix}1 & \alpha_s\\ \alpha_l & 1\end{bmatrix},\qquad \mathbf{V}=\begin{bmatrix}1 & \alpha_s\\ -\alpha_l & -1\end{bmatrix}.$$

Also define the following complex vector

$$\mathbf{f}_m(z;t)=(F_m(z;t),\ G_m(z;t))^{\mathrm{T}},$$

where $z=\eta_1+i\eta_2$. Then, the traction free condition on the crack faces, Eq. (24), can be rewritten as

$$\mathbf{P}\mathbf{f}_m''^{\pm}(\eta_1;t)+\mathbf{Q}\bar{\mathbf{f}}_m''^{\mp}(\eta_1;t)=\mathbf{o},\quad \forall\eta_1<0, \tag{27}$$

and the continuity condition of the displacement and traction ahead of the crack tip, Eqs. (25) and (26), become

$$\left.\begin{aligned}\mathbf{P}\mathbf{f}_m''^{+}(\eta_1;t) + \mathbf{Q}\bar{\mathbf{f}}_m''^{-}(\eta_1;t) - \mathbf{P}\mathbf{f}_m''^{-}(\eta_1;t) - \mathbf{Q}\bar{\mathbf{f}}_m''^{+}(\eta_1;t) = \mathbf{o}\\ \mathbf{U}\mathbf{f}_m'^{+}(\eta_1;t) + \mathbf{V}\bar{\mathbf{f}}_m'^{-}(\eta_1;t) - \mathbf{U}\mathbf{f}_m'^{-}(\eta_1;t) - \mathbf{V}\bar{\mathbf{f}}_m'^{+}(\eta_1;t) = \mathbf{o}\end{aligned}\right\}, \quad \forall\eta_1 > 0. \tag{28}$$

The continuity conditions in (28) can be rearranged as

$$\left.\begin{aligned}\mathbf{P}\mathbf{f}_m''^{+}(\eta_1;t) - \mathbf{Q}\bar{\mathbf{f}}_m''^{+}(\eta_1;t) = \mathbf{P}\mathbf{f}_m''^{-}(\eta_1;t) - \mathbf{Q}\bar{\mathbf{f}}_m''^{-}(\eta_1;t)\\ \mathbf{U}\mathbf{f}_m'^{+}(\eta_1;t) - \mathbf{V}\bar{\mathbf{f}}_m'^{+}(\eta_1;t) = \mathbf{U}\mathbf{f}_m'^{-}(\eta_1;t) - \mathbf{V}\bar{\mathbf{f}}_m'^{-}(\eta_1;t)\end{aligned}\right\}, \quad \forall\eta_1 > 0. \tag{29}$$

From above equations, we may define two new functions by

$$\begin{aligned}\boldsymbol{\kappa}_m(z;t) &= \mathbf{P}\mathbf{f}_m''(z;t) - \mathbf{Q}\bar{\mathbf{f}}_m''(z;t)\\ \boldsymbol{\theta}_m(z;t) &= \mathbf{U}\mathbf{f}_m'(z;t) - \mathbf{V}\bar{\mathbf{f}}_m'(z;t).\end{aligned} \tag{30}$$

$\boldsymbol{\kappa}_m(z;t)$ and $\boldsymbol{\theta}_m(z;t)$ are analytic functions throughout the z-plane except along the cut occupied by the crack. From Eq. (30), it can be seen immediately that Eq. (28) is satisfied identically. So, the issue now is to find the analytic functions $\boldsymbol{\kappa}_m(z;t)$ and $\boldsymbol{\theta}_m(z;t)$.

Solve for $\mathbf{f}_m''(z;t)$ and $\bar{\mathbf{f}}_m''(z;t)$ from Eq. (30) to get

$$\begin{aligned}\mathbf{f}_m''(z;t) &= \mathbf{P}^{-1}\mathbf{H}^{-1}\{\boldsymbol{\theta}_m'(z;t) - \overset{*}{\mathbf{L}}\boldsymbol{\kappa}_m(z;t)\}\\ \bar{\mathbf{f}}_m''(z;t) &= \mathbf{Q}^{-1}\mathbf{H}^{-1}\{\boldsymbol{\theta}_m'(z;t) - \mathbf{L}\boldsymbol{\kappa}_m(z;t)\},\end{aligned} \tag{31}$$

where

$$\mathbf{L} = \mathbf{U}\mathbf{P}^{-1}, \qquad \overset{*}{\mathbf{L}} = \mathbf{V}\mathbf{Q}^{-1}, \qquad \mathbf{H} = \mathbf{L} - \overset{*}{\mathbf{L}}.$$

Here, we have assumed that the inverse matrices $\mathbf{P}^{-1}$ and $\mathbf{Q}^{-1}$ exist. Notice that the determinants of $\mathbf{P}$ and $\mathbf{Q}$ are both equal to $D(v)$, where

$$D(v) = 4\alpha_l\alpha_s - (1 + \alpha_s^2)^2.$$

Therefore, we exclude the situation where the crack propagates with the Rayleigh wave speed of the elastic material. This ensures the existence of $\mathbf{P}^{-1}$ and $\mathbf{Q}^{-1}$.

Substituting the expressions in Eq. (31) into the traction free conditions on the crack faces, (27), and notice that $\mathbf{H} \neq \mathbf{o}$ for $v(t) \neq 0$, we get

$$\left.\begin{aligned}\boldsymbol{\theta}_m'^{+}(\eta_1;t) - \overset{*}{\mathbf{L}}\boldsymbol{\kappa}_m^{+}(\eta_1;t) + \boldsymbol{\theta}_m'^{-}(\eta_1;t) - \mathbf{L}\boldsymbol{\kappa}_m^{-}(\eta_1;t) = \mathbf{o}\\ \boldsymbol{\theta}_m'^{-}(\eta_1;t) - \overset{*}{\mathbf{L}}\boldsymbol{\kappa}_m^{-}(\eta_1;t) + \boldsymbol{\theta}_m'^{+}(\eta_1;t) - \mathbf{L}\boldsymbol{\kappa}_m^{+}(\eta_1;t) = \mathbf{o}\end{aligned}\right\}, \quad \forall\eta_1 < 0. \tag{32}$$

Subtracting the second equation in (32) from the first one, we obtain

$$\boldsymbol{\kappa}_m^+(\eta_1;t) - \boldsymbol{\kappa}_m^-(\eta_1;t) = \mathbf{o}, \qquad \forall \eta_1 < 0, \tag{33}$$

which implies that $\boldsymbol{\kappa}_m(z;t)$ is continuous across the negative real axis except at the crack tip and therefore $\boldsymbol{\kappa}_m(z;t)$ is analytic in the entire complex plane except at $z = 0$. However, the condition of bounded displacement requires that $|\boldsymbol{\kappa}_m(z;t)| = O(|z|^\alpha)$ for some $\alpha > -1$, as $|z| \to 0$. So that any singularity of $\boldsymbol{\kappa}_m(z;t)$ at the crack tip is removable. Therefore, $\boldsymbol{\kappa}_m(z;t)$ is an entire function. Now, both equations in (32) become

$$\boldsymbol{\theta}_m'^+(\eta;t) + \boldsymbol{\theta}_m'^-(\eta_1;t) = (\mathbf{L} + \overset{*}{\mathbf{L}})\boldsymbol{\kappa}_m(\eta_1;t), \quad \forall \eta_1 < 0, \tag{34}$$

where

$$\boldsymbol{\kappa}_m(\eta_1;t) = \boldsymbol{\kappa}_m^+(\eta_1;t) = \boldsymbol{\kappa}_m^-(\eta_1;t).$$

Equation (34) constitutes a Riemann–Hilbert problem. Its solution $\boldsymbol{\theta}_m'(z;t)$ is analytic in the cut plane. Along the cut, $\boldsymbol{\theta}_m'(z;t)$ satisfies Eq. (34) for some arbitrary entire function $\boldsymbol{\kappa}_m(z;t)$. Also, from the requirement of bounded displacements at the crack tip, as $|z| \to 0$,

$$|\boldsymbol{\theta}_m'(z;t)| = O(|z|^\alpha), \tag{35}$$

for some $\alpha > -1$.

In Eq. (34), the solution $\boldsymbol{\theta}_m'(z;t)$ is composed by two parts, the homogeneous solution $\mathring{\boldsymbol{\theta}}_m'(z;t)$, and the particular solution $\hat{\boldsymbol{\theta}}_m'(z;t)$. The homogeneous solution $\mathring{\boldsymbol{\theta}}_m'(z;t)$ can be obtained as [17]

$$\mathring{\boldsymbol{\theta}}_m'(z;t) = z^{-1/2}\mathring{\mathbf{a}}_m(z;t), \tag{36}$$

where $\mathring{\mathbf{a}}_m(z;t)$ is an arbitrary entire function. The particular solution $\hat{\boldsymbol{\theta}}_m'(z;t)$ can also be easily constructed by considering that $\boldsymbol{\kappa}_m(z;t)$ is an entire function and by using the identity theorem for analytic functions. The particular solution is given by

$$\hat{\boldsymbol{\theta}}_m'(z;t) = \tfrac{1}{2}(\mathbf{L} + \overset{*}{\mathbf{L}})\boldsymbol{\kappa}_m(z;t). \tag{37}$$

The final solution for $\boldsymbol{\theta}_m'(z;t)$ is then

$$\boldsymbol{\theta}_m'(z;t) = z^{-1/2}\mathring{\mathbf{a}}_m(z;t) + \tfrac{1}{2}(\mathbf{L} + \overset{*}{\mathbf{L}})\boldsymbol{\kappa}_m(z;t). \tag{38}$$

Substituting Eq. (38) into (31), we have

$$\mathbf{f}''_m(z;t) = \mathbf{P}^{-1}\{z^{-1/2}\mathbf{a}_m(z;t) + \mathbf{b}_m(z;t)\}$$
$$\bar{\mathbf{f}}''_m(z;t) = \mathbf{Q}^{-1}\{z^{-1/2}\mathbf{a}_m(z;t) - \mathbf{b}_m(z;t)\}, \tag{39}$$

where

$$\mathbf{a}_m(z;t) = \mathbf{H}^{-1}\mathring{\mathbf{a}}_m(z;t), \qquad \mathbf{b}_m(z;t) = \tfrac{1}{2}\boldsymbol{\kappa}_m(z;t).$$

Suppose that $\mathbf{a}_m(z;t)$ and $\mathbf{b}_m(z;t)$ have components like

$$\mathbf{a}_m(z;t) = (a_m^{(1)}(z;t),\ a_m^{(2)}(z;t))^{\mathrm{T}}$$
$$\mathbf{b}_m(z;t) = (b_m^{(1)}(z;t),\ b_m^{(2)}(z;t))^{\mathrm{T}}.$$

By comparing the conjugate of $\bar{\mathbf{f}}''_m(z;t)$ with $\mathbf{f}''_m(z;t)$ in Eq. (39), and by using the fact that

$$\mathbf{PQ}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

we obtain

$$a_m^{(1)}(z;t) - \bar{a}_m^{(1)}(z;t) = 0, \qquad a_m^{(2)}(z;t) + \bar{a}_m^{(2)}(z;t) = 0$$
$$b_m^{(1)}(z;t) + \bar{b}_m^{(1)}(z;t) = 0, \qquad b_m^{(2)}(z;t) - \bar{b}_m^{(2)}(z;t) = 0. \tag{40}$$

As a result of above relations, the four undetermined entire functions $a_m^{(\alpha)}(z;t)$ and $b_m^{(\alpha)}(z;t)$ ($\alpha = 1, 2$) can be reduced to two by defining

$$A_m(z;t) = \tfrac{1}{2}\{a_m^{(1)}(z;t) + \bar{a}_m^{(1)}(z;t) + a_m^{(2)}(z;t) - \bar{a}_m^{(2)}(z;t)\}$$
$$B_m(z;t) = \tfrac{1}{2}\{b_m^{(1)}(z;t) - \bar{b}_m^{(1)}(z;t) + b_m^{(2)}(z;t) + \bar{b}_m^{(2)}(z;t)\}. \tag{41}$$

Now we can express the function $\mathbf{f}''_m(z;t)$ in terms of the two undetermined entire functions $A_m(z;t)$ and $B_m(z;t)$ by

$$\mathbf{f}''_m(z;t) = \tfrac{1}{2}z^{-1/2}\{\mathbf{P}^{-1}\boldsymbol{\eta}A_m(z;t) + \mathbf{Q}^{-1}\boldsymbol{\eta}\bar{A}_m(z;t)$$
$$+ \tfrac{1}{2}\{\mathbf{P}^{-1}\boldsymbol{\eta}B_m(z;t) - \mathbf{Q}^{-1}\boldsymbol{\eta}\bar{B}_m(z;t)\}, \tag{42}$$

where $\boldsymbol{\eta} = (1, 1)^T$. Since $A_m(z;t)$ and $B_m(z;t)$ are entire functions, they can be expanded into Taylor series. Define

$$
\begin{aligned}
\tfrac{1}{2}\{A_m(z;t) + \bar{A}_m(z;t)\} &= -\sum_{n=0}^{\infty} A_{Im}^{(n)}(t) z^n \\
\tfrac{1}{2}\{A_m(z;t) - \bar{A}_m(z;t)\} &= -i\sum_{n=0}^{\infty} A_{IIm}^{(n)}(t) z^n \\
\tfrac{1}{2}\{B_m(z;t) + \bar{B}_m(z;t)\} &= -\sum_{n=0}^{\infty} B_{Im}^{(n)}(t) z^n \\
\tfrac{1}{2}\{B_m(z;t) - \bar{B}_m(z;t)\} &= -i\sum_{n=0}^{\infty} B_{IIm}^{(n)}(t) z^n ,
\end{aligned}
\tag{43}
$$

where $A_{Im}^{(n)}(t)$, $A_{IIm}^{(n)}(t)$, $B_{Im}^{(n)}(t)$, and $B_{IIm}^{(n)}(t)$ are real functions of time t. Also, by considering the properties of our asymptotic expansion, (15) and (16), for $m = 0$ and 1, we have

$$
\begin{aligned}
F_0''(z_l;t) = {} & \sum_{n=0}^{\infty} \left\{ \frac{1+\alpha_s^2}{\mu D(v)} A_{I0}^{(n)}(t) z_l^{n-1/2} - \frac{2\alpha_s}{\mu D(v)} B_{I0}^{(n)}(t) z_l^n \right\} \\
& - i \sum_{n=0}^{\infty} \left\{ \frac{2\alpha_s}{\mu D(v)} A_{II0}^{(n)}(t) z_l^{n-1/2} - \frac{1+\alpha_s^2}{\mu D(v)} B_{II0}^{(n)}(t) z_l^n \right\} \\
G_0''(z_s;t) = {} & -\sum_{n=0}^{\infty} \left\{ \frac{2\alpha_l}{\mu D(v)} A_{I0}^{(n)}(t) z_s^{n-1/2} - \frac{1+\alpha_s^2}{\mu D(v)} B_{I0}^{(n)}(t) z_s^n \right\} \\
& + i \sum_{n=0}^{\infty} \left\{ \frac{1+\alpha_s^2}{\mu D(v)} A_{II0}^{(n)}(t) z_s^{n-1/2} - \frac{2\alpha_l}{\mu D(v)} B_{II0}^{(n)}(t) z_s^n \right\}
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
F_1''(z_l;t) = {} & -\sum_{n=0}^{\infty} \left\{ \frac{2\alpha_s}{\mu D(v)} B_{I1}^{(n)}(t) z_l^n - \frac{1+\alpha_s^2}{\mu D(v)} A_{I1}^{(n)}(t) z_l^{n+1/2} \right\} \\
& + i \sum_{n=0}^{\infty} \left\{ \frac{1+\alpha_s^2}{\mu D(v)} B_{II1}^{(n)}(t) z_l^n - \frac{2\alpha_s}{\mu D(v)} A_{II1}^{(n)}(t) z_l^{n+1/2} \right\} \\
G_1''(z_s;t) = {} & \sum_{n=0}^{\infty} \left\{ \frac{1+\alpha_s^2}{\mu D(v)} B_{I1}^{(n)}(t) z_s^n - \frac{2\alpha_l}{\mu D(v)} A_{I1}^{(n)}(t) z_s^{n+1/2} \right\} \\
& - i \sum_{n=0}^{\infty} \left\{ \frac{2\alpha_l}{\mu D(v)} B_{II1}^{(n)}(t) z_s^n - \frac{1+\alpha_s^2}{\mu D(v)} A_{II1}^{(n)}(t) z_s^{n+1/2} \right\}.
\end{aligned}
\tag{45}
$$

By integrating above expressions with respect to the corresponding argument z_l or z_s, we can obtain the final expressions of the complex displacement potentials $F_m(z_l;t)$ and $G_m(z_s;t)$ for $m = 0$ and 1. If the crack propagates along a straight path, Eq. (44) actually has provided the complete solution for the steady state problem under mixed mode loading conditions, while all coefficients do not depend on time. It can be shown that the coefficients of the most singular terms, $A_{I0}^{(0)}(t)$ and $A_{II0}^{(0)}(t)$, can be rewritten as

$$A_{I0}^{(0)}(t) = \frac{K_I^d(t)}{\sqrt{2\pi}}, \qquad A_{II0}^{(0)}(t) = \frac{K_{II}^d(t)}{\sqrt{2\pi}}, \tag{46}$$

where $K_I^d(t)$ and $K_{II}^d(t)$ are the mode-I and mode-II dynamic stress intensity factors at the moving crack tip, respectively.

3.2. Solutions for $\phi_m(\eta_1, \eta_2, t)$ and $\psi_m(\eta_1, \eta_2, t)$ for $m = 2$

For $m = 2$, the equations of motion (17) and (18) are coupled. They take the form,

$$\begin{aligned}
&\phi_{2,11}(\eta_1, \eta_2, t) + \frac{1}{\alpha_l^2}\phi_{2,22}(\eta_1, \eta_2, t) \\
&\quad = -\frac{2\sqrt{v}}{\alpha_l^2 c_s^2}\{\sqrt{v}\phi_{0,1}\}_{,t} + \frac{(1-\alpha_l^2)k}{\alpha_l^2}\{\phi_{0,2} + 2\eta_1\phi_{0,12} - 2\eta_2\phi_{0,11}\} \\
&\psi_{2,11}(\eta_1, \eta_2, t) + \frac{1}{\alpha_s^2}\psi_{2,22}(\eta_1, \eta_2, t) \\
&\quad = -\frac{2\sqrt{v}}{\alpha_s^2 c_s^2}\{\sqrt{v}\psi_{0,1}\}_{,t} + \frac{(1-\alpha_s^2)k}{\alpha_s^2}\{\psi_{0,2} + 2\eta_1\psi_{0,12} - 2\eta_2\psi_{0,11}\}.
\end{aligned} \tag{47}$$

where $\phi_0(\eta_1, \eta_2, t)$ and $\psi_0(\eta_1, \eta_2, t)$ have been given in the previous section.

In order to obtain the next most singular term in $\phi_2(\eta_1, \eta_2, t)$ and $\psi_2(\eta_1, \eta_2, t)$, we should only consider the most singular terms in $\phi_0(\eta_1, \eta_2, t)$ and $\psi_0(\eta_1, \eta_2, t)$. As a result, $\phi_0(\eta_1, \eta_2, t)$ and $\psi_0(\eta_1, \eta_2, t)$ can be written as

$$\phi_0(\eta_1, \eta_2, t) = \mathrm{Re}\{K_l(t) z_l^{3/2}\}, \qquad \psi_0(\eta_1, \eta_2, t) = \mathrm{Im}\{K_s(t) z_s^{3/2}\}, \tag{48}$$

where

$$\begin{aligned}
K_l(t) &= \frac{4(1+\alpha_s^2)}{3\sqrt{2\pi}\,\mu D(v)} K_I^d(t) - i\frac{8\alpha_s}{3\sqrt{2\pi}\,\mu D(v)} K_{II}^d(t) \\
K_s(t) &= -\frac{8\alpha_l}{3\sqrt{2\pi}\,\mu D(v)} K_I^d(t) + i\frac{4(1+\alpha_s^2)}{3\sqrt{2\pi}\,\mu D(v)} K_{II}^d(t).
\end{aligned}$$

Substituting Eq. (84) into (47), we get

$$\phi_{2,11} + \frac{1}{\alpha_l^2}\phi_{2,22} = \mathrm{Re}\{R_l(t)z_l^{1/2} - S_l(t)\bar{z}_l z_l^{-1/2}\}$$

$$\psi_{2,11} + \frac{1}{\alpha_s^2}\psi_{2,22} = \mathrm{Im}\{R_s(t)z_s^{1/2} - S_s(t)\bar{z}_s z_s^{-1/2}\}, \tag{49}$$

where

$$R_{l,s}(t) = D_{l,s}^1\{K_{l,s}(t)\} + \tfrac{1}{2}B_{l,s}(t) + M_{l,s}(t)$$

$$S_{l,s}(t) = \tfrac{1}{2}B_{l,s}(t) + N_{l,s}(t),$$

and

$$D_{l,s}^1\{K_{l,s}(t)\} = -\frac{3\sqrt{v}}{\alpha_{l,s}^2 c_{l,s}^2}\frac{\mathrm{d}}{\mathrm{d}t}\{\sqrt{v}\,K_{l,s}(t)\}$$

$$B_{l,s}(t) = \frac{3v^2\dot{v}}{2\alpha_{l,s}^4 c_{l,s}^4}K_{l,s}(t)$$

$$M_{l,s}(t) = i\,\frac{3(1-\alpha_{l,s}^2)(1+3\alpha_{l,s}^2)}{4\alpha_{l,s}^3}K_{l,s}(t)k(t)$$

$$N_{l,s}(t) = i\,\frac{3(1-\alpha_{l,s}^2)^2}{4\alpha_{l,s}^3}K_{l,s}(t)k(t).$$

The most general solutions to Eq. (49) are

$$\phi_2(\eta_1, \eta_2, t) = \mathrm{Re}\{F_2(z_l;t) + \bar{z}_l f_l(z_l;t) + \bar{z}_l^2 g_l(z_l;t)\}$$

$$\psi_2(\eta_1, \eta_2, t) = \mathrm{Im}\{G_2(z_s;t) + \bar{z}_s f_s(z_s;t) + \bar{z}_s^2 g_s(z_s;t)\}, \tag{50}$$

where

$$f_{l,s}(z_{l,s};t) = \tfrac{1}{6}R_{l,s}(t)z_{l,s}^{3/2}, \qquad g_{l,s}(z_{l,s};t) = -\tfrac{1}{4}S_{l,s}(t)z_{l,s}^{1/2},$$

and $F_2(z_l;t)$, $G_2(z_s;t)$ are two analytic functions in the corresponding cut planes. It can be seen that $f_{l,s}(z_{l,s};t)$ and $g_{l,s}(z_{l,s};t)$ are totally determined by the solutions $\phi_0(\eta_1, \eta_2, t)$ and $\psi_0(\eta_1, \eta_2, t)$, and they depend on $\dot{K}_I^d(t)$, $\dot{K}_{II}^d(t)$, $\dot{v}(t)$, and for a crack propagates along a curved path, they also depend on the curvature of the path at the crack tip, $k(t)$.

Associated with $\phi_2(\eta_1, \eta_2, t)$ and $\psi_2(\eta_1, \eta_2, t)$, given in Eq. (50), the corresponding components of displacement can be expressed as

$$
\begin{aligned}
u_1^{(2)} = \operatorname{Re}\{&F_2'(z_l;t) + \alpha_s G_2'(z_s;t) \\
&+ [\bar{z}_l f_l'(z_l;t) + \bar{z}_l^2 g_l'(z_l;t) + f_l(z_l;t) + 2\bar{z}_l g_l(z_l;t)] \\
&+ \alpha_s[\bar{z}_s f_s'(z_s;t) + \bar{z}_s^2 g_s'(z_s;t) - f_s(z_s;t) - 2\bar{z}_s g_s(z_s;t)\} \\
u_2^{(2)} = -\operatorname{Im}\{&\alpha_l F_2'(z_l;t) + G_2'(z_s;t) \\
&+ \alpha_l[\bar{z}_l f_l'(z_l;t) + \bar{z}_l^2(z_l;t) - f_l(z_l;t) - 2\bar{z}_l g_l(z_l;t)] \\
&+ [\bar{z}_s fs'(z_s;t) + \bar{z}_s^2 g_s'(z_s;t) + f_s(z_s;t) + 2\bar{z}_s g_s(z_s;t)\}.
\end{aligned}
\tag{51}
$$

The stress components are

$$
\begin{aligned}
\sigma_{11}^{(2)} = \mu \operatorname{Re}\Bigg\{&(1 + 2\alpha_l^2 - \alpha_s^2)F_2''(z_l;t) + 2\alpha_s G_2''(z_s;t) \\
&+ (1 + 2\alpha_l^2 - \alpha_s^2)[\bar{z}_l f_l''(z_l;t) + \bar{z}_l^2 g_l''(z_l;t) + 2g_l(z_l;t)] \\
&+ 2\left[(1 - \alpha_s^2) + \frac{2\alpha_l^2(\alpha_l^2 - \alpha_s^2)}{1 - \alpha_l^2}\right][f_l'(z_l;t) + 2\bar{z}_l g_l'(z_l;t)] \\
&+ 2\alpha_s[\bar{z}_s f_s''(z_s;t) + \bar{z}_s^2 g_s''(z_s;t) - 2g_s(z_s;t)]\Bigg\},
\end{aligned}
\tag{52}
$$

$$
\begin{aligned}
\sigma_{22}^{(2)} = -\mu \operatorname{Re}\Bigg\{&(1 + \alpha_s^2)F_2''(z_l;t) + 2\alpha_s G_2''(z_s;t) \\
&+ (1 + \alpha_s^2)[\bar{z}_l f_l''(z_l;t) + \bar{z}_l^2 g_l''(z_l;t) + 2g_l(z_l;t)] \\
&+ 2\left[(1 - \alpha_s^2) - \frac{2(\alpha_l^2 - \alpha_s^2)}{1 - \alpha_l^2}\right][f_l'(z_l;t) + 2\bar{z}_l g_l'(z_l;t)] \\
&+ 2\alpha_s[\bar{z}_s f_s''(z_s;t) + \bar{z}_s^2 g_s''(z_s;t) - 2g_s(z_s;t)]\Bigg\},
\end{aligned}
\tag{53}
$$

and

$$
\begin{aligned}
\sigma_{12}^{(2)} = -\mu \operatorname{Im}\{&2\alpha_l F_2''(z_l;t) + (1 + \alpha_s^2)G_2''(z_s;t) \\
&+ 2\alpha_l[\bar{z}_l f_l''(z_l;t) + \bar{z}_l^2 g_l''(z_l;t) - 2g_l(z_l;t)] \\
&+ (1 + \alpha_s^2)[\bar{z}_s f_s''(z_s;t) + \bar{z}_s^2 g_s''(z_s;t) + 2g_s(z_s;t)] \\
&+ 2(1 - \alpha_s^2)[f_s'(z_s;t) + 2\bar{z}_s^2 g_s'(z_s;t)]\}.
\end{aligned}
\tag{54}
$$

To produce a more compact form of the above expressions, one needs to define the following quantities,

$$\overset{*}{\mathbf{P}} = \begin{bmatrix} \mu(1 + \alpha_s^2 & -2\mu\alpha_s \\ -2\mu\alpha_l & \mu(1 + \alpha_s^2) \end{bmatrix}, \qquad \overset{*}{\mathbf{Q}} = \begin{bmatrix} \mu(1 + \alpha_s^2) & -2\mu\alpha_s \\ 2\mu\alpha_l & -\mu(1 + \alpha_s^2) \end{bmatrix},$$

$$\overset{*}{\mathbf{U}} = \begin{bmatrix} 1 & -\alpha_s \\ -\alpha_l & 1 \end{bmatrix}, \qquad \overset{*}{\mathbf{V}} = \begin{bmatrix} 1 & -\alpha_s \\ \alpha_l & -1 \end{bmatrix},$$

and

$$\mathbf{M} = \begin{bmatrix} \mu\left\{(1 - \alpha_s^2) - \dfrac{2(\alpha_l^2 - \alpha_s^2)}{1 - \alpha_l^2}\right\} & 0 \\ 0 & \mu(1 - \alpha_s^2) \end{bmatrix},$$

$$\mathbf{N} = \begin{bmatrix} \mu\left\{(1 - \alpha_s^2) - \dfrac{2(\alpha_l^2 - \alpha_s^2)}{1 - \alpha_l^2}\right\} & 0 \\ 0 & -\mu(1 - \alpha_s^2) \end{bmatrix}.$$

Also, let

$$\mathbf{f}_2(z;t) = (F_2(z;t),\ G_2(z;t))^{\mathrm{T}}$$
$$\mathbf{f}(z;t) = (f_l(z;t),\ f_s(z;t))^{\mathrm{T}}$$
$$\mathbf{g}(z;t) = (g_l(z;t),\ g_s(z;t))^{\mathrm{T}}.$$

As in the procedure we used to obtain the complex displacement potentials for $m = 0$ and 1, we may define two new functions $\boldsymbol{\kappa}_2(z;t)$ and $\boldsymbol{\theta}_2(z;t)$, so that the continuity conditions ahead of the crack tip are satisfied identically, as follows,

$$\begin{aligned} \boldsymbol{\kappa}_2(z;t) = {} & \mathbf{P}\{\mathbf{f}_2''(z;t) + z\mathbf{f}''(z;t) + z^2\mathbf{g}''(z;t)\} \\ & -\mathbf{Q}\{\bar{\mathbf{f}}_2''(z;t) + z\bar{\mathbf{f}}''(z;t) + z^2\bar{\mathbf{g}}''(z;t)\} \\ & +2\mathbf{M}\{\mathbf{f}'(z;t) + 2z\mathbf{g}'(z;t)\} - 2\mathbf{N}\{\bar{\mathbf{f}}'(z;t) + 2z\bar{\mathbf{g}}'(z;t)\} \\ & +2\overset{*}{\mathbf{P}}\mathbf{g}(z;t) - 2\overset{*}{\mathbf{Q}}\bar{\mathbf{g}}(z;t), \end{aligned} \tag{55}$$

and

$$\begin{aligned} \boldsymbol{\theta}_2(z;t) = {} & \mathbf{U}\{\mathbf{f}_2'(z;t) + z\mathbf{f}'(z;t) + z^2\mathbf{g}'(z;t)\} \\ & -\mathbf{V}\{\bar{\mathbf{f}}_2'(z;t) + z\bar{\mathbf{f}}'(z;t) + z^2\bar{\mathbf{g}}'(z;t)\} \\ & +\overset{*}{\mathbf{U}}\{\mathbf{f}(z;t) + 2z\mathbf{g}(z;t)\} - \overset{*}{\mathbf{V}}\{\bar{\mathbf{f}}(z;t) + 2z\bar{\mathbf{g}}(z;t)\}, \end{aligned} \tag{56}$$

where $\boldsymbol{\kappa}_2(z;t)$ and $\boldsymbol{\theta}_2(z;t)$ are analytic in the cut plane. In order to keep our notation short, define a new quantity,

$$\mathbf{g}_2(z;t) = \mathbf{f}_2''(z;t) + z\mathbf{f}''(z;t) + z^2\mathbf{g}''(z;t) + 2\mathbf{P}^{-1}\mathbf{M}\{\mathbf{f}'(z;t) + 2z\mathbf{g}'(z;t)\} + 2\mathbf{P}^{-1}\overset{*}{\mathbf{P}}\mathbf{g}(z;t).$$

Now, the expressions (55) and (56) can be simplified to

$$\begin{aligned}\boldsymbol{\kappa}_2(z;t) &= \mathbf{P}\mathbf{g}_2(z;t) - \mathbf{Q}\bar{\mathbf{g}}_2(z;t)\\ \boldsymbol{\theta}_2'(z;t) &= \mathbf{U}\mathbf{g}_2(z;t) - \mathbf{V}\bar{\mathbf{g}}_2(z;t) - \{\mathbf{q}(z;t) - \overset{*}{\mathbf{q}}(z;t)\},\end{aligned} \tag{57}$$

where

$$\begin{aligned}\mathbf{q}(z;t) &= 2(\mathbf{L}\mathbf{M} - \mathbf{I})\{\mathbf{f}'(z;t) + 2z\mathbf{g}'(z;t)\} + 2(\mathbf{L}\overset{*}{\mathbf{P}} - \overset{*}{\mathbf{U}})\mathbf{g}(z;t)\\ \overset{*}{\mathbf{q}}(z;t) &= 2(\overset{*}{\mathbf{L}}\mathbf{N} - \mathbf{J})\{\mathbf{f}'(z;t) + 2z\mathbf{g}'(z;t)\} + 2(\overset{*}{\mathbf{L}}\overset{*}{\mathbf{Q}} - \overset{*}{\mathbf{V}})\mathbf{g}(z;t),\end{aligned} \tag{58}$$

and

$$\mathbf{I} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

By solving Eq. (57), we obtain

$$\begin{aligned}\mathbf{g}_2(z;t) &= \mathbf{P}^{-1}\mathbf{H}^{-1}\{\boldsymbol{\theta}_2'(z;t) - \overset{*}{\mathbf{L}}\boldsymbol{\kappa}_2(z;t) + \mathbf{q}(z;t) - \overset{*}{\mathbf{q}}(z;t)\}\\ \bar{\mathbf{g}}_2(z;t) &= \mathbf{Q}^{-1}\mathbf{H}^{-1}\{\boldsymbol{\theta}_2'(z;t) - \mathbf{L}\boldsymbol{\kappa}_2(z;t) + \mathbf{q}(z;t) - \overset{*}{\mathbf{q}}(z;t)\}.\end{aligned} \tag{59}$$

It can be seen that the above equation is very similar to Eq. (31), except the term $\mathbf{q}(z;t) - \overset{*}{\mathbf{q}}(z;t)$ which is totally determined by the solution for $m = 0$. On the other hand, it can also be shown that the traction free condition on the crack faces reduces to

$$\mathbf{P}\mathbf{g}_2^{\pm}(\eta_1;t) + \mathbf{Q}\bar{\mathbf{g}}_2^{\mp}(\eta_1;t) = \mathbf{0}, \quad \forall \eta_1 < 0. \tag{60}$$

Substituting Eq. (59) into the above boundary conditions, and similar to the procedure used in the case for $m = 0$ and 1, one can show that $\boldsymbol{\kappa}_2(z;t)$ is an entire function. Meanwhile, conditions (15) and (16) require that $\boldsymbol{\kappa}_2(z;t)| = O(|z|)$, as $|z| \to 0$. Finally, Eq. (60) become

$$\boldsymbol{\theta}_2'^{+}(\eta_1;t) + \boldsymbol{\theta}_2'^{-}(\eta_1;t) = (\mathbf{L} + \overset{*}{\mathbf{L}})\boldsymbol{\kappa}_2(\eta_1;t) + \hat{\boldsymbol{\kappa}}(\eta_1;t), \quad \forall \eta_1 < 0. \tag{61}$$

where

$$\hat{\boldsymbol{\kappa}}(\eta_1;t) = -\{\mathbf{q}^{+}(\eta_1;t) + \mathbf{q}^{-}(\eta_1;t) - \bar{\overset{*}{\mathbf{q}}}{}^{+}(\eta_1;t) - \bar{\overset{*}{\mathbf{q}}}{}^{-}(\eta_1;t)\}.$$

By substituting the expressions of $\mathbf{q}(z;t)$ and $\overset{*}{\mathbf{q}}(z;t)$ into above relations, we get

$$\hat{\boldsymbol{\kappa}}(\eta_1;t) = \mathbf{o}, \quad \forall \eta_1 < 0.$$

Therefore, the equation that $\boldsymbol{\theta}_2'(z;t)$ should satisfy, is

$$\boldsymbol{\theta}_2'^{+}(\eta_1;t) + \boldsymbol{\theta}_2'^{-}(\eta_1;t) = (\mathbf{L} + \overset{*}{\mathbf{L}})\boldsymbol{\kappa}_2(\eta_1;t), \quad \forall \eta_1 < 0. \tag{62}$$

This is exactly the same as Eq. (34). One basic difference, however, is that from the properties of our asymptotic expansion, (15) and (16), as $|z| \to 0$,

$$|\boldsymbol{\theta}_2'(z;t)| = O(|z|^{\alpha}), \tag{63}$$

for some $\alpha > 0$ (recall that before $\alpha > -1$). As a result, the solution of $\boldsymbol{\theta}_2'(z;t)$ will be

$$\boldsymbol{\theta}_2'(z;t) = z^{1/2}\mathring{\mathbf{a}}_2(z;t) + \tfrac{1}{2}(\mathbf{L} + \overset{*}{\mathbf{L}})\boldsymbol{\kappa}_2(z;t), \tag{64}$$

where $\mathring{\mathbf{a}}_2(z;t)$ is an arbitrary entire function.

In constructing the solution for $\mathbf{g}_2(z;t)$, only the leading term in (64) is considered. This is consistent with the fact that Eq. (48) contains only leading terms of the solution for $m = 0$. The final solution for $\mathbf{g}_2(z;t)$ is therefore

$$\mathbf{g}_2(z;t) = \tfrac{1}{2}\{\mathbf{P}^{-1}\boldsymbol{\eta}A_2(t) + \mathbf{Q}^{-1}\eta\bar{A}_2(t)\}z^{1/2}, \tag{65}$$

for some undetermined complex function of time $A_2(t)$.

Our final target is to find the function $\mathbf{f}_2(z;t)$. After some manipulations, we obtain

$$\begin{aligned}\mathbf{f}_2(z;t) &= \tfrac{2}{15}\{\mathbf{P}^{-1}\boldsymbol{\eta}A_2(t) + \mathbf{Q}^{-1}\boldsymbol{\eta}\bar{A}_2(t)\}z^{5/2}\\ &\quad + \tfrac{4}{15}\{\boldsymbol{\Gamma}\boldsymbol{\gamma}(t) - \boldsymbol{\Omega}\boldsymbol{\omega}(t)\}z^{5/2},\end{aligned} \tag{66}$$

where

$$\boldsymbol{\gamma}(t) = (R_l(t),\ R_s(t))^{\mathsf{T}}, \quad \boldsymbol{\omega}(t) = (S_l(t),\ S_s(t))^{\mathsf{T}},$$

and

$$\mathbf{\Gamma} = \begin{bmatrix} \dfrac{(1+\alpha_s^2)m_l}{D(v)} - \dfrac{1}{8} & -\dfrac{2\alpha_s m_s}{D(v)} \\ -\dfrac{2\alpha_l m_l}{D(v)} & \dfrac{(1+\alpha_s^2)m_s}{D(v)} - \dfrac{1}{8} \end{bmatrix},$$

$$\mathbf{\Omega} = \begin{bmatrix} \dfrac{(1+\alpha_s^2)m_l}{D(v)} + \dfrac{\overset{*}{D}(v)}{D(v)} + \dfrac{1}{16} & -\dfrac{2\alpha_s m_s}{D(v)} - \dfrac{2\alpha_s(1+\alpha_s^2)}{D(v)} \\ -\dfrac{2\alpha_l m_l}{D(v)} - \dfrac{2\alpha_l(1+\alpha_s^2)}{D(v)} & \dfrac{(1+\alpha_s^2)m_s}{D(v)} + \dfrac{\overset{*}{D}(v)}{D(v)} + \dfrac{1}{16} \end{bmatrix}.$$

In the matrices above, the quantities m_l, m_s, and $\overset{*}{D}(v)$ are given by

$$m_l = \frac{1}{2}\left\{(1-\alpha_s^2) - \frac{2(\alpha_l^2 - \alpha_s^2)}{1-\alpha_l^2}\right\}$$

$$m_s = \frac{1}{2}\{1-\alpha_s^2\}$$

$$\overset{*}{D}(v) = 4\alpha_l\alpha_s + (1+\alpha_s^2)^2. \tag{67}$$

In this section, we have provided a procedure which allows us to investigate higher order transient effects systematically. By imposing the boundary conditions along the crack faces and the continuity conditions ahead of the crack tip on the complex potentials, the problem can be recast into the Riemann–Hilbert methodology. In summarizing, in the unscaled physical plane, let

$$z_{l,s} = \xi_1 + i\alpha_{l,s}\xi_2, \qquad z = \xi_1 + i\xi_2,$$

and

$$\mathbf{f}_m(z;t) = (F_m(z;t), G_m(z;t))^{\mathrm{T}}, \quad m = 0, 1, 2.$$

Then

$$\begin{aligned} \mathbf{f}_0(z;t) &= \tfrac{1}{2}\{\mathbf{P}^{-1}\boldsymbol{\eta}A_0(t) + \mathbf{Q}^{-1}\boldsymbol{\eta}\bar{A}_0(t)\}z^{3/2} \\ \mathbf{f}_1(z;t) &= \tfrac{1}{2}\{\mathbf{P}^{-1}\boldsymbol{\eta}A_1(t) - \mathbf{Q}^{-1}\boldsymbol{\eta}\bar{A}_1(t)\}z^{2} \\ \mathbf{f}_2(z;t) &= \tfrac{1}{2}\{\mathbf{P}^{-1}\boldsymbol{\eta}A_2(t) + \mathbf{Q}^{-1}\boldsymbol{\eta}\bar{A}_2(t)\}z^{5/2}. \\ &\quad + \tfrac{4}{15}\{\mathbf{\Gamma}\boldsymbol{\gamma}(t) - \mathbf{\Omega}\boldsymbol{\omega}(t)\}z^{5/2}. \end{aligned} \tag{68}$$

Notice that since $A_m(t)$ $(m = 0, 1, 2)$ are arbitrary functions of time, we have redefined them in Eq. (68). Specifically, $A_0(t)$ is related to the so called mixed-mode dynamic stress intensity factors, $K_I^d(t)$ and $K_{II}^d(t)$, by

$$A_0(t) = -\frac{4}{3\sqrt{2\pi}} \{K_I^d(t) + iK_{II}^d(t)\}. \tag{69}$$

The corresponding displacement potentials $\phi_m(\xi_1, \xi_2, t)$ and $\psi_m(\xi_1, \xi_2, t)$, will be given by (21) for $m = 0$ and 1, and (50) for $m = 2$, respectively. Finally,

$$\phi(\xi_1, \xi_2, t) = \sum_{m=0}^{2} \phi_m(\xi_1, \xi_2, t) + O(r_l^3)$$

$$\psi(\xi_1, \xi_2, t) = \sum_{m=0}^{2} \psi_m(\xi_1, \xi_2, t) + O(r_s^3), \tag{70}$$

where $r_{l,s} = (\xi_1^2 + \alpha_{l,s}^2\xi_2^2)^{1/2}$.

Equation (70) provides the first three terms of the asymptotic expansion for the two displacement potentials $\phi(\xi_1, \xi_2, t)$ and $\psi(\xi_1, \xi_2, t)$ for a dynamic crack propagating non-uniformly along an arbitrary path. This expansion is based on an assumption that the fields are indeed two dimensional right up to the crack tip. It is observed that the first two terms are the same as those obtained under the steady state mixed-mode condition and the crack path is straight, except here the coefficients $A_0(t)$ and $A_2(t)$ are arbitrary functions of time and the crack tip speed takes the instantaneous value at time t. However, generally speaking, under the mixed-mode loading conditions, the crack will no longer propagate along a straight path and it is commonly believed that the crack will seek the direction where locally the mode-I condition prevails. So the crack will propagate along a curved trajectory for the most general loading conditions. Even if the loading condition is mode-I, and the crack does propagate along a straight path, when the crack tip speed is sufficiently high, the moving crack will lose its stability and deviate from the original straight path to propagate along a curve. The third term, or the higher order term in (70), takes into account the recent past history of the mixed-mode stress intensity factors and crack motion. This term involves the time derivatives of the dynamic stress intensity factors, $K_I^d(t)$ and $K_{II}^d(t)$, and crack tip speed $v(t)$. It also involves the crack tip curvature $k(t)$ as well. From Eqs. (17) and (18), it can be seen that as we go further to the terms with $m > 2$, higher order time derivatives of $K_I^d(t)$, $K_{II}^d(t)$, and crack tip speed $v(t)$ must be involved, so is the time derivative of the crack tip curvature $k(t)$. The procedure discussed in this section is constructive and it can be repeated to any order.

4. The asymptotic elastodynamic field around a non-uniformly propagating crack tip

For the planar deformation of a homogeneous, isotropic, linearly elastic material, the ordered array $[u_\alpha, \varepsilon_{\alpha\beta}, \sigma_{\alpha\beta}]$, α, $\beta \in \{1, 2\}$, is said to be an elastodynamic state in the absence of body force density, if the following conditions are satisfied

$$\left.\begin{aligned} \varepsilon_{\alpha\beta} &= \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \\ \sigma_{\alpha\beta} &= 2\mu\varepsilon_{\alpha\beta} + \lambda\varepsilon_{\gamma\gamma}\delta_{\alpha\beta} \\ \sigma_{\alpha\beta,\beta} &= \rho\ddot{u}_\alpha, \end{aligned}\right\}, \quad \alpha, \beta \in \{1, 2\}, \tag{71}$$

where ρ is the mass density and λ, μ are Lamé constants of the material. In addition, the field quantities $u_\alpha, \varepsilon_{\alpha\beta}$, and $\sigma_{\alpha\beta}$ must satisfy the smoothness requirements outlined in Wheeler and Sternberg [18].

In the Cartesian coordinate system (ξ_1, ξ_2), let $\phi_m(\xi_1, \xi_2, t)$ and $\psi_m(\xi_1, \xi_2, t)$ be solutions of Eqs. (17) and (18), $m = 0, 1, 2, \ldots$, such that

$$\left.\begin{aligned} \frac{\phi_{m+n}(\xi_1, \xi_2, t)}{\phi_m(\xi_1, \xi_2, t)} &\to 0 \\ \frac{\psi_{m+n}(\xi_1, \xi_2, t)}{\psi_m(\xi_1, \xi_2, t)} &\to 0 \end{aligned}\right\}, \quad \text{as } r = \sqrt{\xi_1^2 + \xi_2^2} \to 0, \quad m = 0, 1, 2, \ldots \tag{72}$$

for any positive integer n. Thus, $\phi_m(\xi_1, \xi_2, t)$ and $\psi_m(\xi_1, \xi_2, t)$ will be two asymptotic sequences as $r = (\xi_1^2 + \xi_2^2)^{1/2} \to 0$. Define $\phi(\xi_1, \xi_2, t)$ and $\psi(\xi_1, \xi_2, t)$ by

$$\phi(\xi_1, \xi_2, t) = \sum_{m=0}^{\infty} \phi_m(\xi_1, \xi_2, t), \qquad \psi(\xi_1, \xi_2, t) = \sum_{m=0}^{\infty} \psi_m(\xi_1, \xi_2, t). \tag{73}$$

Then, the array $[u_\alpha, \varepsilon_{\alpha\beta}, \sigma_{\alpha\beta}]$, α, $\beta \in \{1, 2\}$, will constitute an *asymptotic* elastodynamic state as $r = (\xi_1^2 + \xi_2^2)^{1/2} \to 0$, if it satisfies

$$\left.\begin{aligned} u_\alpha &= \phi_{,\alpha} + e_{\alpha\beta}\psi_{,\beta} \\ \varepsilon_{\alpha\beta} &= \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \\ \sigma_{\alpha\beta} &= 2\mu\varepsilon_{\alpha\beta} + \lambda\varepsilon_{\gamma\gamma}\delta_{\alpha\beta} \end{aligned}\right\}, \quad \alpha, \beta \in \{1, 2\}. \tag{74}$$

Let the two displacement potentials be given by (73), where each term of the asymptotic series is the solution which has been discussed in the previous section. The asymptotic elastodynamic state near the tip of a non-uniformly propagating crack along an arbitrary path, can therefore be obtained from relations (74). For its importance in the experimental investigation, we provide here the asymptotic expression of the stress components around the moving crack tip by using the constitutive relation (3). With respect to the ξ_1-axis, we can observe that the two displacement potentials $\phi(\xi_1, \xi_2, t)$ and $\psi(\xi_1, \xi_2, t)$ are composed by two parts, $\phi^{(I)}(\xi_1, \xi_2, t)$, $\psi^{(I)}(\xi_1, \xi_2, t)$ and $\phi^{(II)}(\xi_1, \xi_2, t)$, $\psi^{(II)}(\xi_1, \xi_2, t)$. This separation is very similar to the decomposition of the deformation field into mode-I, or the symmetric part, and mode-II, or the asymmetric part, when we deal with near tip deformation of a straight moving crack and claim that the deformation field is the superposition of these two modes. As a result of this separation, in writing the expression of the stress components $\sigma_{\alpha\beta}(\xi_1, \xi_2, t)$, $\alpha, \beta \in \{1, 2\}$, we may also separate $\sigma_{\alpha\beta}(\xi_1, \xi_2, t)$ into two parts, the part $\sigma_{\alpha\beta}^{(I)}(\xi_1, \xi_2, t)$ associated with the symmetric deformation and the part $\sigma_{\alpha\beta}^{(II)}(\xi_1, \xi_2, t)$ associated with the asymmetric deformation. Meanwhile, define the scaled polar coordinates $(r_{l,s}, \theta_{l,s})$ by

$$\tau_{l,s} = \sqrt{\xi_1^2 + \alpha_{l,s}^2 \xi_2^2}, \qquad \theta_{l,s} = \tan^{-1} \frac{\alpha_{l,s}\xi_2}{\xi_1}.$$

Then, we have

$$\sigma_{\alpha\beta}(\xi_1, \xi_2, t) = \sigma_{\alpha\beta}^{(I)}(\xi_1, \xi_2, t) + \sigma_{\alpha\beta}^{(II)}(\xi_1, \xi_2, t), \quad \alpha, \beta \in \{1, 2\}. \tag{75}$$

The stress components associated with the symmetric deformation in Eq. (75) are

$$\frac{\sigma_{11}^{(I)}}{\mu} = \frac{K_I^d(t)}{\mu\sqrt{2\pi}} \left\{ \frac{(1 + 2\alpha_l^2 - \alpha_s^2)(1 + \alpha_s^2)}{D(v)} r_l^{-1/2} \cos\frac{\theta_l}{2} - \frac{4\alpha_l\alpha_s}{D(v)} r^{-1/2} \cos\frac{\theta_s}{2} \right\}$$
$$+ \frac{4\alpha_s(\alpha_l^2 - \alpha_s^2)}{\mu D(v)} \mathrm{Re}\{A_1(t)\} + \mathrm{Re} \left\{ \left[-\frac{15(1 + 2\alpha_l^2 - \alpha_s^2)(1 + \alpha_s^2)}{4\mu D(v)} A_2(t) \right.\right.$$
$$+ (1 + 2\alpha_l^2 - \alpha_s^2) f_l(t) + \left(\frac{1 - \alpha_s^2}{2} + \frac{\alpha_l^2(\alpha_l^2 - \alpha_s^2)}{1 - \alpha_l^2} \right) R_l(t) \Bigg] \cos\frac{\theta_l}{2}$$
$$+ \left[\frac{1 + 2\alpha_l^2 - \alpha_s^2}{8} R_l(t) - \left(\frac{1 - \alpha_s^2}{2} + \frac{\alpha_l^2(\alpha_l^2 - \alpha_s^2)}{1 - \alpha_l^2} \right) S_l(t) \right] \cos\frac{3\theta_l}{2}$$

$$+\frac{1+2\alpha_l^2-\alpha_s^2}{16} S_l(t) \cos\frac{7\theta_l}{2}\bigg\} r_l^{1/2}$$

$$+2\alpha_s \operatorname{Re}\left\{\left[\frac{15\alpha_l}{2\mu D(v)} A_2(t)+g_s(t)\right]\cos\frac{\theta_s}{2}+\frac{1}{8} R_s(t)\cos\frac{3\theta_s}{2}\right.$$

$$\left.+\frac{1}{16} S_s(t)\cos\frac{7\theta_s}{2}\right\} r_s^{1/2}+O(r_{l,s}), \tag{76}$$

$$\frac{\sigma_{22}^{(I)}}{\mu}=\frac{K_I^d(t)}{\mu\sqrt{2\pi}}\left\{-\frac{(1+\alpha_s^2)^2}{D(v)} r_l^{-1/2}\cos\frac{\theta_l}{2}+\frac{4\alpha_l\alpha_s}{D(v)} r_s^{-1/2}\cos\frac{\theta_s}{2}\right\}$$

$$-\operatorname{Re}\left\{\left[-\frac{15(1+\alpha_s^2)^2}{4\mu D(v)} A_2(t)+(1+\alpha_s^2)f_l(t)\right.\right.$$

$$\left.+\left(\frac{1-\alpha_s^2}{2}-\frac{\alpha_l^2-\alpha_s^2}{1-\alpha_l^2}\right) R_l(t)\right]\cos\frac{\theta_l}{2}$$

$$+\left[\frac{1+\alpha_s^2}{8} R_l(t)-\left(\frac{1-\alpha_s^2}{2}-\frac{\alpha_l^2-\alpha_s^2}{1-\alpha_l^2}\right) S_l(t)\right]\cos\frac{3\theta_l}{2}$$

$$\left.+\frac{1+\alpha_s^2}{16} S_l(t)\cos\frac{7\theta_l}{2}\right\} r_l^{1/2}$$

$$-2\alpha_s \operatorname{Re}\left\{\left[\frac{15\alpha_l}{2\mu D(v)} A_2(t)+g_s(t)\right]\cos\frac{\theta_s}{2}+\frac{1}{8} R_s(t)\cos\frac{3\theta_s}{2}\right.$$

$$\left.+\frac{1}{16} S_s(t)\cos\frac{7\theta_s}{2}\right\} r_s^{1/2}+O(r_{l,s}), \tag{77}$$

and

$$\frac{\sigma_{12}^{(I)}}{\mu}=\frac{K_I^d(t)}{\mu\sqrt{2\pi}}\frac{2\alpha_l(1+\alpha_s^2)}{D(v)}\left\{r_l^{-1/2}\sin\frac{\theta_l}{2}-r_s^{-1/2}\sin\frac{\theta_s}{2}\right\}$$

$$-2\alpha_l \operatorname{Re}\left\{\left[-\frac{15(1+\alpha_s^2)}{4\mu D(v)} A_2(t)+g_l(t)\right]\sin\frac{\theta_l}{2}-\frac{1}{8} R_l(t)\sin\frac{3\theta_l}{2}\right.$$

$$\left.-\frac{1}{16} S_l(t)\sin\frac{7\theta_l}{2}\right\} r_l^{1/2}$$

$$-\operatorname{Re}\left\{\left[\frac{15\alpha_l(1+\alpha_s^2)}{2\mu D(v)} A_2(t)+(1+\alpha_s^2)f_s(t)+\frac{1-\alpha_s^2}{2} R_s(t)\right]\sin\frac{\theta_s}{2}\right.$$

$$\left.-\left[\frac{1+\alpha_s^2}{8} R_s(t)-\frac{1-\alpha_s^2}{2} S_s(t)\right]\sin\frac{3\theta_s}{2}-\frac{1+\alpha_s^2}{16} S_s(t)\sin\frac{7\theta_s}{2}\right\} r_s^{1/2}$$

$$+O(r_{l,s})$$

The stress components associated with the asymmetric deformation in Eq. (75) are

$$
\begin{aligned}
\frac{\sigma^{(II)}}{\mu} = &-\frac{2\alpha_s K_{II}^d(t)}{\mu\sqrt{2\pi}}\left\{\frac{1+2\alpha_l^2-\alpha_s^2}{D(v)}\, r_l^{-1/2}\sin\frac{\theta_l}{2} - \frac{1+\alpha_s^2}{D(v)}\, r_s^{-1/2}\sin\frac{\theta_s}{2}\right\} \\
&+\operatorname{Im}\left\{\left[-\frac{15\alpha_s(1+2\alpha_l^2-\alpha_s^2)}{2\mu D(v)}A_2(t) - (1+2\alpha_l^2-\alpha_s^2)f_l(t)\right.\right. \\
&\left.-\left(\frac{1-\alpha_s^2}{2}+\frac{\alpha_l^2(\alpha_l^2-\alpha_s^2)}{1-\alpha_l^2}\right)R_l(t)\right]\sin\frac{\theta_l}{2} \\
&+\left[\frac{1+2\alpha_l^2-\alpha_s^2}{8}R_l(t) - \left(\frac{1-\alpha_s^2}{2}+\frac{\alpha_l^2(\alpha_l^2-\alpha_s^2)}{1-\alpha_l^2}\right)S_l(t)\right]\sin\frac{3\theta_l}{2} \\
&\left.+\frac{1+2\alpha_l^2-\alpha_s^2}{16}S_l(t)\sin\frac{7\theta_l}{2}\right\}r_l^{1/2} \\
&+2\alpha_s\operatorname{Im}\left\{\left[\frac{15(1+\alpha_s^2)}{4\mu D(v)}A_2(t) - g_s(t)\right]\sin\frac{\theta_s}{2}+\frac{1}{8}R_s(t)\sin\frac{3\theta_s}{2}\right. \\
&\left.+\frac{1}{16}S_s(t)\sin\frac{7\theta_s}{2}\right\}r_s^{1/2} + O(r_{l,s}),
\end{aligned}
\tag{79}
$$

$$
\begin{aligned}
\frac{\sigma_{22}^{(II)}}{\mu} = &\frac{K_{II}^d(t)}{\mu\sqrt{2\pi}}\frac{2\alpha_s(1+\alpha_s^2)}{D(v)}\left\{r_l^{-1/2}\sin\frac{\theta_l}{2} - r_s^{-1/2}\sin\frac{\theta_s}{2}\right\} \\
&-\operatorname{Im}\left\{\left[-\frac{15\alpha_s(1+\alpha_s^2)}{2\mu D(v)}A_2(t) - (1+\alpha_s^2)f_l(t)\right.\right. \\
&\left.-\left(\frac{1-\alpha_s^2}{2}-\frac{\alpha_l^2-\alpha_s^2}{1-\alpha_l^2}\right)R_l(t)\right]\sin\frac{\theta_l}{2} \\
&+\left[\frac{1+\alpha_s^2}{8}R_l(t) - \left(\frac{1-\alpha_s^2}{2}-\frac{\alpha_l^2-\alpha_s^2}{1-\alpha_l^2}\right)S_l(t)\right]\sin\frac{3\theta_l}{2} \\
&\left.+\frac{1+\alpha_s^2}{16}S_l(t)\sin\frac{7\theta_l}{2}\right\}r_l^{1/2} \\
&-2\alpha_s\operatorname{Im}\left\{\left[\frac{15(1+\alpha_s^2)}{4\mu D(v)}A_2(t) - g_s(t)\right]\sin\frac{\theta_s}{2}+\frac{1}{8}R_s(t)\sin\frac{3\theta_s}{2}\right. \\
&\left.+\frac{1}{16}S_s(t)\sin\frac{7\theta_s}{2}\right\}r_s^{1/2} + O(r_{l,s}),
\end{aligned}
\tag{80}
$$

and

$$\frac{\sigma_{12}^{(II)}}{\mu} = \frac{K_{II}^d(t)}{\mu\sqrt{2\pi}} \left\{ \frac{4\alpha_l\alpha_s}{D(v)} r_l^{-1/2} \cos\frac{\theta_l}{2} - \frac{(1+\alpha_s^2)^2}{D(v)} r_s^{-1/2} \cos\frac{\theta_s}{2} \right\}$$

$$-2\alpha_l \operatorname{Im}\left\{\left[\frac{15\alpha_s}{2\mu D(v)} A_2(t) + g_l(t)\right] \cos\frac{\theta_l}{2} + \frac{1}{8} R_l(t) \cos\frac{3\theta_l}{2}\right.$$

$$\left. + \frac{1}{16} S_l(t) \cos\frac{7\theta_l}{2}\right\} r_l^{1/2}$$

$$-\operatorname{Im}\left\{\left[-\frac{15(1+\alpha_s^2)^2}{4\mu D(v)} A_2(t) + \frac{1-\alpha_s^2}{2} R_s(t)\right] \cos\frac{\theta_s}{2}\right.$$

$$\left. + \left[\frac{1+\alpha_s^2}{8} R_s(t) - \frac{1-\alpha_s^2}{2} S_s(t)\right] \cos\frac{3\theta_s}{2} + \frac{1+\alpha_s^2}{16} S_s(t) \cos\frac{7\theta_s}{2}\right\} r_s^{1/2}$$

$$+ O(r_{l,s}). \tag{81}$$

In the above expressions for components of stress, $K_I^d(t)$ and $K_{II}^d(t)$ are the mixed-mode dynamic stress intensity factors. The complex functions $f_{l,s}(t)$ and $g_{l,s}(t)$ are given by

$$f_l(t) = \left(\frac{(1+\alpha_s^2)m_l}{D(v)} - \frac{1}{8}\right) R_l(t) - \left(\frac{(1+\alpha_s^2)m_l}{D(v)} + \frac{\overset{*}{D}(v)}{D(v)} + \frac{9}{16}\right) S_l(t)$$

$$- \frac{2\alpha_s m_s}{D(v)} R_s(t) + \left(\frac{2\alpha_s m_s}{D(v)} + \frac{2\alpha_s(1+\alpha_s^2)}{D(v)}\right) S_s(t)$$

$$f_s(t) = -\frac{2\alpha_l m_l}{D(v)} R_l(t) + \left(\frac{2\alpha_l m_l}{D(v)} + \frac{2\alpha_l(1+\alpha_s^2)}{D(v)}\right) S_l(t)$$

$$+ \left(\frac{(1+\alpha_s^2)m_s}{D(v)} - \frac{1}{8}\right) R_s(t) - \left(\frac{(1+\alpha_s^2)m_s}{D(v)} + \frac{\overset{*}{D}(v)}{D(v)} + \frac{9}{16}\right) S_s(t)$$

$$g_l(t) = \left(\frac{(1+\alpha_s^2)m_l}{D(v)} - \frac{1}{8}\right) R_l(t) - \left(\frac{(1+\alpha_s^2)m_l}{D(v)} + \frac{\overset{*}{D}(v)}{D(v)} - \frac{7}{16}\right) S_l(t)$$

$$- \frac{2\alpha_s m_s}{D(v)} R_s(t) + \left(\frac{2\alpha_s m_s}{D(v)} + \frac{2\alpha_s(1+\alpha_s^2)}{D(v)}\right) S_s(t)$$

$$g_s(t) = -\frac{2\alpha_l m_l}{D(v)} R_l(t) + \left(\frac{2\alpha_l m_l}{D(v)} + \frac{2\alpha_l(1+\alpha_s^2)}{D(v)}\right) S_l(t)$$

$$+ \left(\frac{(1+\alpha_s^2)m_s}{D(v)} - \frac{1}{8}\right) R_s(t) - \left(\frac{(1+\alpha_s^2)m_s}{D(v)} + \frac{\overset{*}{D}(v)}{D(v)} - \frac{7}{16}\right) S_s(t), \tag{82}$$

where m_l, m_s, and $\overset{*}{D}(v)$ are functions of the crack tip speed and are given in Eq. (67). Also, more explicitly, we can express the quantities $S_{l,s}(t)$ and $R_{l,s}(t)$ in terms of the mixed-mode dynamic stress intensity factors $K_I^d(t)$ and $K_{II}^d(t)$, the time derivative of the crack tip speed $\dot{v}(t)$, and the curvature of the trajectory at the crack tip $k(t)$ as follows

$$S_l(t) = \left\{\frac{v^2(1+\alpha_s^2)}{\mu\sqrt{2\pi}\,D(v)\alpha_l^4 c_l^4} K_I^d(t)\dot{v}(t) + \frac{2\alpha_s(1-\alpha_l^2)^2}{\mu\sqrt{2\pi}\,D(v)\alpha_l^3} K_{II}^d(t)k(t)\right\}$$
$$-i\left\{\frac{2v^2\alpha_s}{\mu\sqrt{2\pi}\,D(v)\alpha_l^4 c_l^4} K_{II}^d(t)\dot{v}(t) - \frac{(1+\alpha_s^2)(1-\alpha_l^2)^2}{\mu\sqrt{2\pi}\,D(v)\alpha_l^3} K_I^d(t)k(t)\right\}$$
$$S_s(t) = -\left\{\frac{2v^2\alpha_l}{\mu\sqrt{2\pi}\,D(v)\alpha_s^4 c_s^4} K_I^d(t)\dot{v}(t) + \frac{(1+\alpha_s^2)(1-\alpha_s^2)^2}{\mu\sqrt{2\pi}\,D(v)\alpha_s^3} K_{II}^d(t)k(t)\right\}$$
$$+i\left\{\frac{v^2(1+\alpha_s^2)}{\mu\sqrt{2\pi}\,D(v)\alpha_s^4 c_s^4} K_{II}^d(t)\dot{v}(t) - \frac{2\alpha_l(1-\alpha_s^2)^2}{\mu\sqrt{2\pi}\,D(v)\alpha_s^3} K_I^d(t)k(t)\right\}, \tag{83}$$

and

$$R_l(t) = -\frac{1}{\mu\sqrt{2\pi}}\left\{\frac{4\sqrt{v}}{\alpha_l^2 c_l^2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\sqrt{v}(1+\alpha_s^2)}{D(v)} K_I^d(t)\right] - \frac{v^2(1+\alpha_s^2)}{D(v)\alpha_l^4 c_l^4} K_I^d(t)\dot{v}(t)\right.$$
$$\left.- \frac{2\alpha_s(1-\alpha_l^2)(1+3\alpha_l^2)}{D(v)\alpha_l^3} K_{II}^d(t)k(t)\right\}$$
$$+i\,\frac{1}{\mu\sqrt{2\pi}}\left\{\frac{8\sqrt{v}}{\alpha_l^2 c_l^2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\sqrt{v}\,\alpha_s}{D(v)} K_{II}^d(t)\right] - \frac{2v^2\alpha_s}{D(v)\alpha_l^4 c_l^4} K_{II}^d(t)\dot{v}(t)\right.$$
$$\left.+ \frac{(1+\alpha_s^2)(1-\alpha_l^2)(1+3\alpha_l^2)}{D(v)\alpha_l^3} K_I^d(t)k(t)\right\}$$
$$R_s(t) = \frac{1}{\mu\sqrt{2\pi}}\left\{\frac{8\sqrt{v}}{\alpha_s^2 c_s^2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\sqrt{v}\,\alpha_l}{D(v)} K_I^d(t)\right] - \frac{2v^2\alpha_l}{D(v)\alpha_s^4 c_s^4} K_I^d(t)\dot{v}(t)\right.$$
$$\left.- \frac{(1+\alpha_s^2)(1-\alpha_s^2)(1+3\alpha_s^2)}{D(v)\alpha_s^3} K_{II}^d(t)k(t)\right\}$$
$$-i\,\frac{i}{\mu\sqrt{2\pi}}\left\{\frac{4\sqrt{v}}{\alpha_s^2 c_s^2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\sqrt{v}(1+\alpha_s^2)}{D(v)} K_{II}^d(t)\right] - \frac{v^2(1+\alpha_s^2)}{D(v)\alpha_s^4 c_s^4} K_{II}^d(t)\dot{v}(t)\right.$$
$$\left.+ \frac{2\alpha_l(1-\alpha_s^2)(1+3\alpha_s^2)}{D(v)\alpha_s^3} K_I^d(t)k(t)\right\}. \tag{84}$$

In the expressions of the components of asymptotic stress field near the moving crack tip, (76) through (81), $R_l(t)$ and $R_s(t)$ depend not only on the mixed-mode dynamic stress intensity factors, $K_I^d(t)$ and $K_{II}^d(t)$, and the crack tip speed, $v(t)$, but also on the time derivatives of these quantities. Meanwhile, $R_l(t)$ and $R_s(t)$ also depend on the trajectory curvature at the crack tip $k(t)$, as shown in Eq. (84). $S_l(t)$ and $S_s(t)$ also have these properties, but they do not depend on the time derivatives of the mixed-mode dynamic stress intensity factors. In most of the experiments, the study of the dynamic crack growth is under mode-I loading conditions and the crack propagates along a straight path. Under these circumstances, $k(t) = 0$, $K_{II}^d(t) = 0$, and all quantities of the form $\text{Im}\{\cdot\}$ disappear, and the deformation field is symmetric. At this point, Eqs. (76) through (78) provide the stress field of a non-uniformly propagating mode-I crack. This is the same as that given by Rosakis et al. [15]. If the crack tip velocity, $v(t)$, is a constant, i.e. $\dot{v}(t) = 0$, and therefore, $S_l(t) = S_s(t) = 0$, we can obtain the asymptotic stress field corresponding to transient crack growth with constant velocity and varying stress intensity factor (see [14]). A classical example of such a transient crack problem is the one analyzed by Broberg [4]. Furthermore, if the time derivative of the dynamic stress intensity factor, $\dot{K}_I^d(t)$, is also zero, so are $R_l(t)$ and $R_s(t)$; we obtain the familiar asymptotic stress field for the steady state situation up to the third term. This is the case considered by Nishioka and Atluri [8], and Dally [9].

5. Discussion and concluding remarks

In this paper, a procedure for obtaining the higher order transient asymptotic representation of the elastodynamic field around the tip of a propagating crack has been developed. The crack propagates transiently along a smooth but otherwise arbitrary path. The material is considered to be homogeneous, isotropic and linearly elastic. The formulation is based on the two displacement potentials, $\phi(\xi_1, \xi_2, t)$ and $\psi(\xi_1, \xi_2, t)$. These two potentials can be expressed in terms of the real and imaginary parts of some complex functions, respectively. By imposing the continuity condition ahead of the crack tip and the traction free boundary condition along the crack faces, the problem can be recast into a Riemann–Hilbert problem. Upon solving the Riemann–Hilbert problem, the two displacement potentials are obtained. Meanwhile, the transient asymptotic representation of the near-tip stress field up to the third term is also provided. The transient effects and the geometrical characteristic of the crack path are included in this analysis.

The general form of the near-tip stress field, Eqs. (76) through (81), exhibits some noteworthy features. First, it is noted that the spatial structure in the *radial direction* of the transient elastodynamic field is the same as that under the steady state conditions. The differences between the results for the transient and the steady state analyses appear in the angular distribution. Secondly, it is observed that the angular distribution for a mixed-mode curving crack is identical to the one corresponding to a mixed-mode crack propagating along a straight line (see [14] and [15]). The information regarding the path curvature $k(t)$ only appears in the coefficients of the expansion. It should be also observed that in the local coordinate system (ξ_1, ξ_2), the two components of the crack tip acceleration vector are $(\dot{v}(t),\ k(t)v^2(t))$ at any instant. The above results, as expected, contain both components of crack tip acceleration in the coefficients of the transient high order terms. In the case of a mode-I crack propagating along a straight line ([14] and [15]), only $\dot{v}(t)$ appears in these coefficients.

Suppose that a crack propagates along a straight path, then $k(t) = 0$ at any time during the propagation. Under this situation, (76) through (81) provide the customarily mixed-mode stress field for a mode-I and a mode-II straight crack, respectively. However, as we have mentioned earlier, under the most general loading conditions, the crack will grow along a curved path. When this happens, even though the deformation field can be separated into a symmetric part and an asymmetric part, the so-called mode-I and mode-II types will be coupled together. This happens since in the higher order contributions to the expression of the stress components associated with the symmetric deformation, the crack tip curvature $k(t)$ always appears as a product with $K_{II}^d(t)$ which is the dynamic stress intensity factor for mode-II. Similarly, in the asymmetric deformation field, the crack tip curvature $k(t)$ always appears in a product with $K_I^d(t)$ where $K_I^d(t)$ is the dynamic stress intensity factor for mode-I. An interesting consequence of the above observation is the following. Suppose that the propagating crack follows the path of $K_{II}^d(t) = 0$ for any time (as proposed by Cotterell and Rice [19] for the quasi-statically growing crack). Then since $k(t)$ will not be zero, the asymmetric part of the stresses will in general survive even if the first term disappears. This may produce an experimental illusion of the existence of a nonzero K_{II}^d, if the experimental data are recorded at some distance away from the crack tip. Rossmanith [20] has studied the rapid curved crack propagation using the dynamic photoelastic method. In the interpretation of his experimental data, Rossmanith used the singular (or the K^d-dominant) stress representation. He found that the values of K_I^d and K_{II}^d depend on the positions of measurement (or depend on fringe order). By using the extrapolation, he observed that as the distance from the moving crack-tip $r \to 0$, or the fringe order tends to infinity, K_I^d tends to a finite value while K_{II}^d becomes infinitely small. Similar experimental observations have been reported by Chona and Shukla [21], and by Shukla and Chona [22], who conducted extensive studies of this phenomenon. They also used dynamic photoelasticity to investigate dynamic crack growth along a curved path. Although their isochromatic data were analyzed on the basis of a mixed-mode, *steady state* higher order expansion, they reported very small values of K_{II}^d (up to 10% of K_I^d at each time). They observed that even if they force K_{II}^d to vanish in their expansion, they can still fit the higher order asymmetric isochromatic patterns by adjusting the coefficient of the third ($r^{1/2}$) term in their expansion. This is exactly the term that in the transient expansion involves the product $k(t)K_I^d(t)$ which appears in Eqs. (79)–(81).

To visualize the above discussions, consider the following special situation. Suppose that at time t, locally, the crack-tip undergoes mode-I deformation which conforms to the criterion proposed by Cotterell and Rice [19]. This criterion requires that the crack will follow the path which will assure that $K_{II}^d = 0$. Meanwhile, assume that at this time, the crack-tip acceleration, the time derivatives of the stress intensity factors, and the higher order coefficient $A_2(t)$ all vanish. In addition, suppose that the crack propagates along a curved path, so that the instantaneous crack-tip curvature is not zero. By using the higher order transient asymptotic stress representations obtained in previous sections, the contours of the following field

$$m(\xi_1, \xi_2) = \frac{1}{2}\cdot\frac{\sigma_1 - \sigma_2}{K_I^d(t)/\sqrt{2\pi R(t)}}, \tag{85}$$

are plotted. In (85), σ_1 and σ_2 are the two principal stresses, and $R(t)$ is such that $k(t) = 1/R(t)$. Notice that the contours of the field $m(\xi_1, \xi_2)$ actually simulate normalized photoelastic fringe patterns surrounding the moving crack-tip. These simulated fringe patterns are given in Fig. 2 where the Poisson's ratio for the solid has been chosen as $\nu = 0.3$ and the crack-tip speed has been set to $v/c_s = 0.35$. Figure 2(a) shows the fringe pattern observed in a relatively large region. The fringe pattern is apparently mixed-mode. However, by recalling that locally, the crack-tip field is pure mode-I, this apparent mixed-mode fringe pattern is due to the "mode-coupling" that stems from the geometrical shape of the curved crack which results in non-zero asymmetric higher order transient contributions. Although in this case, the tangential acceleration of the crack-tip, $\dot{v}(t)$, is zero, the instantaneous angular acceleration is finite and equal to $k(t)v^2(t)$. Figure 2(b) represents a view of the same fringe pattern taken much closer to the crack-tip than the view in Fig. 2(a). Figure 2(b) clearly shows that the near-tip field is indeed symmetric. The above observations suggest that the accurate measurement of the dynamic stress intensity factors at a moving crack-tip requires that data points should be chosen either very close to the crack-tip, so that K^d-dominance is valid and can be used, or otherwise a complete higher order transient asymptotic representation should be used to interpret the measurements.

In conclusion we should point out that the field presented above contains,

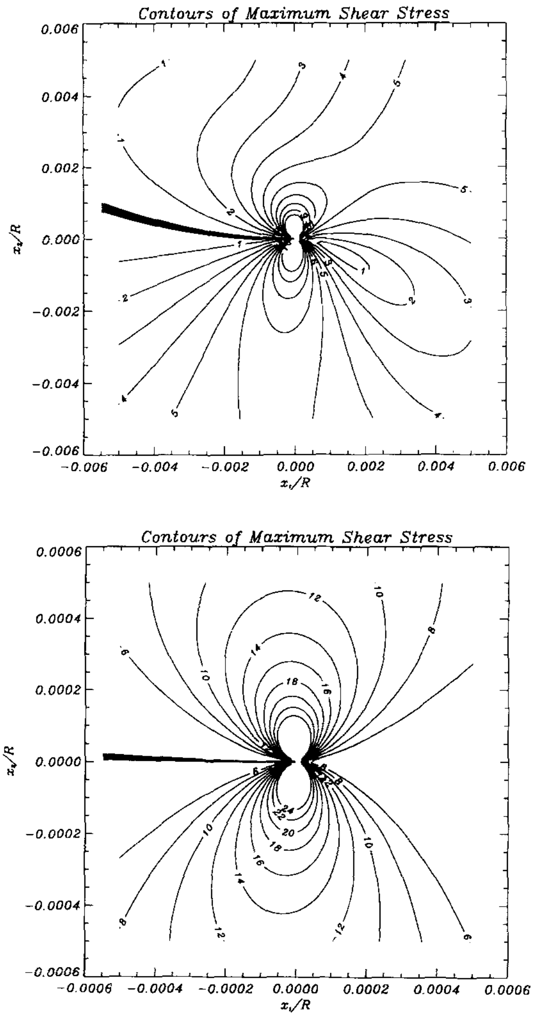

Fig. 2. Simulated photoelastic fringe patterns surrounding the tip of a crack propagating along a curved path, (a) larger observation region, (b) observation in the region very close to the crack tip.

for the first time, both the transient and the geometric features of crack growth. In this sense, it is hoped that it may prove useful in studying crack-tip kinking or curving even in laboratory situations where specimen size and geometry make the existence of transients unavoidable.

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